ANALYTIC EQUIVALENCE OF NORMAL CROSSING FUNCTIONS ON A REAL ANALYTIC MANIFOLD

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Abstract. By Hironaka Desingularization Theorem, any real analytic function has only normal crossing singularities after a modification. We focus on the analytic equivalence of such functions with only normal crossing singularities. We prove that for such functions C^∞ right equivalence implies analytic equivalence. We prove moreover that the cardinality of the set of equivalence classes is zero or countable.

1. Introduction

The classification of real analytic functions is a difficult but fascinating topic in singularity theory. In this paper, we put our interest on real analytic functions with only normal crossing singularities. This case is of fundamental importance since any analytic function becomes one with only normal crossing singularities after a finite sequence of blowings-up along smooth center by Hironaka Desingularization Theorem [Hi]. Our goal is to establish the cardinality of the set of equivalence classes of analytic functions with only normal crossing singularities under analytic equivalence (theorem 3.2).

Our first main result is theorem 3.1,(1) which asserts that C^∞ right equivalent real analytic functions with only normal crossing singularities are automatically analytically right equivalent. Its proof consists in a careful use of Cartan Theorems A and B and Oka Theorem in order to use integration along analytic vector fields to produce analytic isomorphisms. Theorem 3.1,(1) is a crucial result in order to deal with cardinality issues, in particular in view to make a reduction to the case of real analytic functions with semialgebraic graph, called Nash functions.

The second main result (theorem 3.2) establishes the cardinality of the set of equivalence classes of real analytic (respectively Nash) functions with only normal crossing singularities on a compact analytic manifold (resp. on a non-necessarily compact Nash manifold) with respect to the analytic (resp. Nash) equivalence. To prove that this cardinality is zero or countable, we first reduce the study to the Nash case by theorem 3.1,(1), then from the non compact to the compact case via Nash sheaf theory, a Nash version of Hironaka Desingularization Theorem and a finer analysis of the normal crossing property on a Nash manifold with corners. Finally Hardt triviality [Ha], Artin-Mazur Theorem (see [S_2]) and Nash Approximation Theorems [S_2], [C-R-S_1] enable to achieve the proof. Note that along the way, we establish (as theorem 3.1,(3)) a C^2 plus semialgebraic version of theorem

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3.1,(1), namely semialgebraically C^2 right equivalent Nash functions with only normal crossing singularities on a Nash manifold are Nash right equivalent (see also theorem 3.1,(2) for a C^2 version).

The paper is organized as follows. In section one, we recall some definitions that are fundamental in the paper, in particular the notion of normal crossing in the case of manifolds with corners. We devote the second section to some preliminaries about real analytic and Nash sheaf theory, that will be crucial tools for the proof of the main theorems, and also a quick overview on the different topologies we will consider on spaces of maps. Third section is dedicated to theorem 3.1,(1) and its proof, and the statement of theorem 3.1,(2) and 3.1,(3), the proof of which we postpone to section five. Actually, even though the statements are very similar, we need to prepare in section four some materials for it. We prove in particular as lemma 4.6 that a normal crossing Nash subset of a non-compact Nash manifold is trivial at infinity, and we compactify in proposition 4.9 a Nash function with only normal crossing singularities. We finally prove theorems 3.1,(2) and 3.1,(3) together with theorem 3.2 in the last section.

In this paper a manifold means a manifold without boundary, analytic manifolds and maps mean real analytic ones unless otherwise specified, and id stands for the identity map.

1.1. Analytic functions with only normal crossing singularities.

Definition 1.1. Let M be an analytic manifold. An analytic function with *only normal crossing singularities* at a point x of M is a function whose germ at x is of the form $\pm x^\alpha (= \pm \prod_{i=1}^n x_i^{\alpha_i})$ up to an additive constant, for some local analytic coordinate system $(x_1, ..., x_n)$ at x and some $\alpha = (\alpha_1, ..., \alpha_n) \neq 0 \in \mathbf{N}^n$. If the function has only normal crossing singularities everywhere, we say that the function has *only normal crossing singularities*.

An analytic subset of an analytic manifold is called *normal crossing* if it is the zero set of an analytic function with only normal crossing singularities. This analytic function is called *defined by* the analytic set. It is not unique. However, the sheaf of $\mathcal{O}$-ideals *defined by* the analytic set is naturally defined and unique. We can naturally stratify a normal crossing analytic subset X into analytic manifolds X_i of dimension i. We call $\{X_i\}$ the *canonical stratification* of X.

1.2. Case of Nash manifolds.

Definition 1.2. A *semialgebraic* set is a subset of a Euclidean space which is described by finitely many equalities and inequalities of polynomial functions. A *Nash manifold* is a C^ω submanifold of a Euclidean space which is semialgebraic. A *Nash function* on a Nash manifold is a C^ω function with semialgebraic graph. A *Nash subset* is the zero set of a Nash function on a Nash manifold. (We call a germ *on* but not *at* X in M to distinguish the case where X is a set from the case of a point.)

We define Nash functions with *only normal crossing singularities*, *normal crossing* Nash subsets of a Nash manifold and the *canonical stratification* of a normal crossing Nash subset similarly to the analytic case.

For elementary properties of Nash manifolds and Nash functions, we refer to [S_2]. As a general flavor, note that Nash functions carry more structure than analytic or semialgebraic ones, and therefore it is useful to dispose of approximation results

In this paper, we will make an intensive use of the two classical approximation theorems by Nash functions, which are quite different in nature. The first one, that we will refer to as Nash Approximation Theorem I, concerns the approximation of semialgebraic C^r maps by Nash maps (see [S_2]). The topology we use in that case is the *semialgebraic C^r topology* on spaces of semialgebraic C^r maps (see subsection 2.3 for an overview about topologies on spaces of maps). Note for instance that, in that topology, a semialgebraic C^1 map between semialgebraic C^1 manifolds close to a semialgebraic C^1 diffeomorphism is a diffeomorphism.

Theorem. *(Nash Approximation Theorem I, [S_2]) Any semialgebraic C^r map between Nash manifolds can be approximated in the semialgebraic C^r topology by a Nash map.*

The other one, say Nash Approximation Theorem II, is a global version of Artin Approximation Theorem on a compact Nash manifold.

Theorem. *(Nash Approximation Theorem II, [C-R-S_1]) Given a Nash function F on $M_1 \times M_2$ for a compact Nash manifold M_1 and a Nash manifold M_2, and an analytic map $f : M_1 \to M_2$ with $F(x, f(x)) = 0$ for $x \in M_1$, then there exists a Nash approximation $\widetilde{f} : M_1 \to M_2$ of f in the C^∞ topology such that $F(x, \widetilde{f}(x)) = 0$ for $x \in M_1$.*

1.3. Manifolds with corners.

Manifolds with corners appear naturally in the study of functions with only normal crossing singularities. A manifold with corners is locally given by charts diffeomorphic to $[0, \infty)^k \times \mathbf{R}^{n-k}$. In this paper we will consider analytic manifold with corners as well as Nash ones. We refer to [K-S] for basics about manifolds with corners.

The definition of the canonical stratification for manifolds can be naturally extended to the boundary of an analytic manifold with corners. However, concerning the notions of singularity and normal crossings, we really need to adapt the definitions.

Definition 1.3. Let f be an analytic function on analytic manifold with corners M. We say f is *singular* at a point x_0 of ∂M if the restriction of f to the stratum of the canonical stratification of ∂M containing x_0 is singular at x_0.

Note in particular that with such a definition, f is singular at points of the stratum of dimension 0 of the canonical stratification of ∂M. This remark will be of importance when dealing with proofs by induction.

To define a function with *only normal crossing singularities* on a manifold with corners M, we need to extend M beyond the corners. More precisely, we can construct an analytic manifold M' which contains M and is of the same dimension by extending a locally finite system of analytic local coordinate neighborhoods of M. We call M' an *analytic manifold extension* of M. In the same way, shrinking M' if necessary we obtain a normal crossing analytic subset X of M' such that $\operatorname{Int} M$ is a union of some connected components of $M' - X$, and f is extended to an analytic function f' on M'.

Definition 1.4. We say that f has *only normal crossing singularities* if $f|_{\operatorname{Int} M}$ does so and if the germ of $(f - f(x_0))\phi$ at each point x_0 of X has only normal crossing singularities, for ϕ an analytic function on M' defined by X.

Now we can define, similarly to the case without corners, a *normal crossing* analytic subset of M and a *normal crossing* sheaf of $\mathcal{O}$-ideals on M.

In the Nash case, we define analogously a *Nash manifold extension* of a Nash manifold with corners, a Nash function with *only normal crossing singularities* on a Nash manifold with corners, a *normal crossing* Nash subset of M and a *normal crossing* sheaf of $\mathcal{N}$-ideals on M.

2. Preliminaries

We dedicate this section to some remainder on real analytic sheaf theory, and prove similar statements in the Nash case that will be of importance in next sections. We finish with an overview of the different topologies on spaces of functions we will make use in that paper, in order to explain the major differences between them.

2.1. Real analytic sheaves.

In this subsection, we deal with the real analytic case of Cartan Theorems A and B, and Oka Theorem.

Let $\mathcal{O}$ and $\mathcal{N}$ denote, respectively, the sheaves of analytic and Nash function germs on an analytic and Nash manifold and let $N(M)$ denote the ring of Nash functions on a Nash manifold M. We write $\mathcal{O}_M$ and $\mathcal{N}_M$ when we emphasize the domain M. Let f_x, X_x, v_x and $\mathcal{M}_x$ denote the germs of f and X at a point x of M, the tangent vector assigned to x by v and the stalk of $\mathcal{M}$ at x for a function f on an analytic (Nash) manifold M, a subset X of M, a vector field v on M and for a sheaf of $\mathcal{O}$- ($\mathcal{N}$-) modules $\mathcal{M}$ on M, respectively. For a compact semialgebraic subset X of a Nash manifold M, let $\mathcal{N}(X)$ denote the germs of Nash functions on X in M, with the topology of the inductive limit space of the topological spaces $N(U)$ endowed with the compact-open C^∞ topology, where U runs through the family of open semialgebraic neighborhoods of X in M. In the same way, we define $\mathcal{O}(X)$ for a compact semianalytic subset X of an analytic manifold M. Here a *semianalytic* subset is a subset whose germ at each point of M is described by finitely many equalities and inequalities of analytic function germs.

Theorem 2.1. *(Cartan Theorem A) Let $\mathcal{M}$ be a coherent sheaf of $\mathcal{O}$-modules on an analytic manifold M. Then for any $x \in M$, the germ $\mathcal{M}_x$ is equal to $H^0(M, \mathcal{M})\mathcal{O}_x$.*

See [G-R] for Cartan Theorems A and B in the complex case and [Ca] for the real case. Next corollary will be useful in this paper. It deals with the case where the number of local generators is uniformly bounded.

Corollary 2.2. *In theorem 2.1, assume that $\mathcal{M}_x$ is generated by a uniform number of elements for any x in M. Then $H^0(M, \mathcal{M})$ is finitely generated as a $H^0(M, \mathcal{O})$-module.*

The corollary is proved in [Co] in the complex case. The real case follows from a complexification of $\mathcal{M}$ as in [Ca].

Theorem 2.3. *(Cartan Theorem B) Let $\mathcal{M}$ be a coherent sheaf of $\mathcal{O}$-modules on an analytic manifold M. Then $H^1(M, \mathcal{M})$ is equal to zero.*

Corollary 2.4. *Let M be an analytic manifold and $X \subset M$ be a global analytic set—the zero set of an analytic function. Let $\mathcal{I}$ be a coherent sheaf of $\mathcal{O}$-ideals on M such that any element of $\mathcal{I}$ vanishes on X. Then any $f \in H^0(M, \mathcal{O}/\mathcal{I})$ can be*

extended to some $F \in C^\omega(M)$, i.e., f is the image of F under the natural map $H^0(M, \mathcal{O}) \to H^0(M, \mathcal{O}/\mathcal{I})$.

If X is normal crossing, we can choose $\mathcal{I}$ to be the function germs vanishing on X. Then $H^0(M, \mathcal{O}/\mathcal{I})$ consists of functions on X whose germs at each point of X are extensible to analytic function germs on M.

Corollary 2.4 follows from theorem 2.3 by considering the exact sequence $0 \to \mathcal{I} \to \mathcal{O} \to \mathcal{I}/\mathcal{O}$.

Theorem 2.5. *(Oka Theorem) Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be coherent sheaves of $\mathcal{O}$-modules on an analytic manifold M, and $h : \mathcal{M}_1 \to \mathcal{M}_2$ be an $\mathcal{O}$-homomorphism. Then* $\operatorname{Ker} h$ *is a coherent sheaf of $\mathcal{O}$-modules.*

See [G-R] in the complex case. The real case follows from complexification [Ca] of $M, \mathcal{M}_1, \mathcal{M}_2$ and h.

2.2. Nash sheaves.

In this subsection M stands for a Nash manifold. A sheaf of $\mathcal{N}$-modules $\mathcal{M}$ on M is called *finite* if for some finite open semialgebraic covering $\{U_i\}$ of M and for each i there exists an exact sequence $\mathcal{N}^{m_i}|_{U_i} \longrightarrow \mathcal{N}^{n_i}|_{U_i} \longrightarrow \mathcal{M}|_{U_i} \longrightarrow 0$ of $\mathcal{N}$-homomorphisms, with $m_i, n_i \in \mathbf{N}$. Non-finite examples are the sheaf of $\mathcal{N}$-ideals $\mathcal{I}$ on $\mathbf{R}$ of germs vanishing on $\mathbf{Z}$ and $\mathcal{N}/\mathcal{I}$.

Theorem 2.6. *(Nash case of Oka Theorem) Let h be an $\mathcal{N}$-homomorphism between finite sheaves of $\mathcal{N}$-modules on a Nash manifold. Then* $\operatorname{Ker} h$ *is finite.*

Proof. Let $h : \mathcal{M}_1 \to \mathcal{M}_2$ be such a homomorphism on a Nash manifold M. There exists a finite open semialgebraic covering $\{U_i\}$ of M such that $\mathcal{M}_j|_{U_i}$, for$j = 1, 2$, satisfy the condition of exact sequence in the definition of a finite sheaf. Therefore it suffices to prove the theorem on each U_i. Now we may assume that $\mathcal{M}_j$, for $j = 1, 2$, are generated by global cross-sections $\alpha_1, ..., \alpha_{n_1}$ and $\beta_1, ..., \beta_{n_2}$, respectively, and there are Nash maps $\gamma_1, ..., \gamma_{n_3} \in N(M)^{n_2}$ which are generators of the kernel of the surjective $\mathcal{N}$-homomorphism $p : \mathcal{N}^{n_2} \supset \mathcal{N}_x^{n_2} \ni (\phi_1, ..., \phi_{n_2}) \to \sum_{i=1}^{n_2} \phi_i \beta_{ix} \in \mathcal{M}_{2x} \subset \mathcal{M}_2,\ x \in M$. Let $\overline{\alpha}_1, ..., \overline{\alpha}_{n_1}$ denote the images of $\alpha_1, ..., \alpha_{n_1}$ in $H^0(M, \mathcal{M}_2)$ under the homomorphism $h_* : H^0(M, \mathcal{M}_1) \to H^0(M, \mathcal{M}_2)$ induced by h.

We prove the theorem by induction on n_2. For $n_2 = 1$, there exist $\hat{\alpha}_1, ..., \hat{\alpha}_{n_1} \in H^0(M, \mathcal{N})$ such that $p_*(\hat{\alpha}_i) = \overline{\alpha}_i$, for $i = 1, ..., n_1$ because the application $p_* : H^0(M, \mathcal{N}) \to H^0(M, \mathcal{M}_2)$ is surjective by theorem 2.8 for $\mathcal{M}_1 = \mathcal{N}$ ([C-R-S$_1$] and [C-S$_3$]). Let $\delta_1, ..., \delta_{n_4} \in N(M)^{n_1}$ be generators of the kernel of the surjective homomorphism $\mathcal{N}^{n_1} \supset \mathcal{N}_x^{n_1} \ni (\phi_1, ..., \phi_{n_1}) \to \sum_{i=1}^{n_1} \phi_i \alpha_{ix} \in \mathcal{M}_{1x} \subset \mathcal{M}_x,\ x \in M$ (we choose the above $\{U_i\}$ so that $\delta_1, ..., \delta_{n_4}$ exist). Multiplying $\alpha_i, \overline{\alpha}_i, \hat{\alpha}_i,\ \gamma_i$ and δ_i by a small positive Nash function, we can assume by the Łojasiewicz inequality that the Nash maps $\hat{\alpha}_i,\ \gamma_i$ and δ_i are bounded. Then by Proposition VI.2.8 in [S$_2$] we can regard M as the interior of a compact Nash manifold possibly with corners $\tilde{M}$ and the maps as the restrictions to M of Nash maps $\tilde{\hat{\alpha}}_i,\ \tilde{\gamma}_i$ and $\tilde{\delta}_i$ on $\tilde{M}$. Replace $\mathcal{M}_1$ and $\mathcal{M}_2$ by the sheaves of $\mathcal{N}$-modules on $\tilde{M}$ given by $\mathcal{N}^{n_1}/(\tilde{\delta}_1, ..., \tilde{\delta}_{n_4})\mathcal{N}^{n_1}$ and $\mathcal{N}/(\tilde{\gamma}_1, ..., \tilde{\gamma}_{n_3})\mathcal{N}$, respectively, and replace $h : \mathcal{M}_1 \to \mathcal{M}_2$ with the $\mathcal{N}$-homomorphism $\tilde{h} : \tilde{M}_1 \to \tilde{M}_2$ defined by

$$\tilde{h}(0, ..., 0, \overset{i}{1}, 0, ..., 0) = \tilde{\hat{\alpha}}_i \mod (\tilde{\gamma}_1, ..., \tilde{\gamma}_{n_3})\mathcal{N},\ i = 1, ..., n_1$$

Then it suffices to see that $\operatorname{Ker}\tilde{h}$ is finite. Hence we assume from the beginning that M is a compact Nash manifold possibly with corners. Then $\operatorname{Ker} h$ is isomorphic to $\mathcal{N}\otimes_{N(M)}\operatorname{Ker} h_*$ by Theorem 5.2 in [C-R-S$_1$]. Hence $\operatorname{Ker} h$ is finite.

Let $n_2>1$ and assume that the theorem holds for n_2-1. Let $\mathcal{M}_0$ denote the sheaf of $\mathcal{N}$-ideals with $\mathcal{M}_{0x}=\{0\}$. Set $\mathcal{M}_3=\mathcal{M}_2/p(\mathcal{N}\times\mathcal{M}_0\times\cdots\times\mathcal{M}_0)$ and let $h_3:\mathcal{M}_1\to\mathcal{M}_3$ denote the composite of h with the projection from $\mathcal{M}_2$ to $\mathcal{M}_3$. Then $\mathcal{M}_3$ is generated by the images $\overline{\beta}_2,...,\overline{\beta}_{n_2}$ of $\beta_2,...,\beta_{n_2}$, and $\gamma'_1,...,\gamma'_{n_3}\in N(M)^{n_2-1}$ are generators of the kernel of the $\mathcal{N}$-homomorphism

$$\mathcal{N}^{n_2-1}\supset\mathcal{N}_x^{n_2-1}\ni(\phi_1,...,\phi_{n_2-1})\to\sum_{i=1}^{n_2-1}\phi_i\overline{\beta}_{i+1x}\in\mathcal{M}_{3x}\subset\mathcal{M}_3,\ x\in M$$

where $\gamma_i=(\gamma_{i,1},...,\gamma_{i,n_2})=(\gamma_{i,1},\gamma'_i)$, for $i=1,...,n_3$. Hence $\mathcal{M}_3$ is finite, and by induction hypothesis $\operatorname{Ker} h_3$ is finite. Consider $h|_{\operatorname{Ker} h_3}:\operatorname{Ker} h_3\to\mathcal{M}_2$. The image is contained in $p(\mathcal{N}\times\mathcal{M}_0\times\cdots\times\mathcal{M}_0)$ which is isomorphic to $(\operatorname{Ker} p\cup\mathcal{N}\times\mathcal{M}_0\times\cdots\times\mathcal{M}_0)/\operatorname{Ker} p$ and then to $\mathcal{N}\times\mathcal{M}_0\times\cdots\times\mathcal{M}_0/(\operatorname{Ker} p\cap\mathcal{N}\times\mathcal{M}_0\times\cdots\mathcal{M}_0)$. Hence we can regard $h|_{\operatorname{Ker} h_3}$ as an $\mathcal{N}$-homomorphism from $\operatorname{Ker} h_3$ to $\mathcal{N}\times\mathcal{M}_0\times\cdots\times\mathcal{M}_0/(\operatorname{Ker} p\cap\mathcal{N}\times\mathcal{M}_0\times\cdots\mathcal{M}_0)$. In order to achieve the proof, we need to prove that $\operatorname{Ker} p\cap\mathcal{N}\times\mathcal{M}_0\times\cdots\mathcal{M}_0$ is finite. Define a sheaf of $\mathcal{N}$-submodules $\mathcal{M}$ of $\mathcal{N}^{n_3}$ on M by

$$\mathcal{M}_x=\{(\phi_1,...,\phi_{n_3})\in\mathcal{N}_x^{n_3}:\sum_{i=1}^{n_3}\phi_i\gamma_{i,jx}=0,\ j=2,...,n_2\}.$$

Then it suffices to see that $\mathcal{M}$ is finite because $\operatorname{Ker} p\cap\mathcal{N}\times\mathcal{M}_0\times\cdots\times\mathcal{M}_0$ is the image of $\mathcal{M}$ under the $\mathcal{N}$-homomorphism : $\mathcal{N}^{n_3}\supset\mathcal{N}_x^{n_3}\ni(\phi_1,...,\phi_{n_3})\to(\sum_{i=1}^{n_3}\phi_i\gamma_{i,1x},0,...,0)\in\mathcal{N}_x\times\{0\}\times\cdots\times\{0\}\subset\mathcal{N}\times\mathcal{M}_0\times\cdots\times\mathcal{M}_0$, $x\in M$. On the other hand, if we define an $\mathcal{N}$-homomorphism $r:\mathcal{N}^{n_3}\to\mathcal{N}^{n_2-1}$ by $r(\phi_1,...,\phi_{n_3})=(\sum_{i=1}^{n_3}\phi_i\gamma_{i,2x},...,\sum_{i=1}^{n_3}\phi_i\gamma_{i,n_2x})$ for $(\phi_1,...,\phi_{n_3})\in\mathcal{N}_x^{n_3}$, $x\in M$, then $\operatorname{Ker} r=\mathcal{M}$. As in the case of $n_2=1$ we reduce the problem to the case where $\gamma_{i,j}$ are bounded and then M is a compact Nash manifold possibly with corners. Then $\operatorname{Ker} r$ is finite by Theorem 5.2 in [C-R-S$_1$].

Thus $\operatorname{Ker} p\cap\mathcal{N}\times\mathcal{M}_0\times\cdots\times\mathcal{M}_0$ is finite. We can regard it as a sheaf of $\mathcal{N}$-ideals. Hence by the result in case of $n_2=1$, $\operatorname{Ker}(h|_{\operatorname{Ker} h_3})=\operatorname{Ker} h$ is finite. □

The following two theorems do not hold for general sheaves of $\mathcal{N}$-modules, [Hu], [B-C-R] and VI.2.10 in [S$_2$]. However, our case is sufficient for the applications we have in mind in this paper.

Theorem 2.7. *(Nash case of Cartan Theorem A) Let $\mathcal{M}$ be a finite sheaf of $\mathcal{N}$-submodules of $\mathcal{N}^n$ on a Nash manifold M for $n>0\in\mathbf{N}$. Then $\mathcal{M}$ is finitely generated by its global cross-sections.*

Proof. We assume that $n>1$ and proceed by induction on n. Let $p:\mathcal{N}^n\to\mathcal{N}^{n-1}$ denote the projection forgetting the first factor, and set $\mathcal{M}_1=\operatorname{Ker} p|_{\mathcal{M}}$ and $\mathcal{M}_2=\operatorname{Im} p|_{\mathcal{M}}$. Then the sequence $0\longrightarrow\mathcal{M}_1\xrightarrow{p_1}\mathcal{M}\xrightarrow{p_2}\mathcal{M}_2\longrightarrow 0$ is exact, we can regard $\mathcal{M}_1$ as a sheaf of $\mathcal{N}$-ideals, which is finite by theorem 2.6, and $\mathcal{M}_2$ is clearly a finite sheaf of $\mathcal{N}$-submodules of $\mathcal{N}^{n-1}$. By induction hypothesis we

$f_1, ..., f_k \in H^0(M, \mathcal{M})$ such that $p_{2*}(f_i) = g_i,\ i = 1, ..., k$ because $f_1, ..., f_k, h_1, ..., h_l$ are generators of $\mathcal{M}$.

Fix i. Since $H^0(M, \mathcal{M}) \subset N(M)^n$ and $H^0(M, \mathcal{M}_2) \subset N(M)^{n-1}$, setting $g_i = (g_{i,2}, ..., g_{i,n})$ we construct $g_{i,1} \in N(M)$ such that $(g_{i,1}, ..., g_{i,n}) \in H^0(M, \mathcal{M})$. For each $x \in M$, the set $\Phi_x = \{\phi \in \mathcal{N}_x : (\phi, g_{ix}) \in \mathcal{M}_x\}$ is a residue class of $\mathcal{N}_x$ modulo $\mathcal{M}_{1x}$, and the correspondence $\Phi : x \to \Phi_x$ is a global cross-section of $\mathcal{N}/\mathcal{M}_1$. Actually, it suffices to check it on each member of a finite open semialgebraic covering of M, we assume that $\mathcal{M}$ is generated by global cross-sections $\alpha_1 = (\alpha_{1,1}, ..., \alpha_{1,n}), ..., \alpha_{k'} = (\alpha_{k',1}, ..., \alpha_{k',n}) \in N(M)^n$. Then $\alpha'_1 = (\alpha_{1,2}, ..., \alpha_{1,n}), ..., \alpha'_{k'} = (\alpha_{k',2}, ..., \alpha_{k',n})$ are also generators of $\mathcal{M}_2$. Let $\mathcal{M}_3$ denote the kernel of the $\mathcal{N}$-homomorphism $\mathcal{N}^{k'+1} \supset \mathcal{N}_x^{k'+1} \ni (\phi_1, ..., \phi_{k'+1}) \to \sum_{j=1}^{k'} \phi_j \alpha'_{jx} - \phi_{k'+1} g_{ix} \in \mathcal{N}_x^{n-1} \subset \mathcal{N}^{n-1},\ x \in M$. Then $\mathcal{M}_3$ is finite by theorem 2.6, and each stalk $\mathcal{M}_{3x}$ contains a germ of the form $(\phi_1, ..., \phi_{k'}, 1)$. Hence refining the covering if necessary, we assume that $\mathcal{M}_3$ is generated by a finite number of global cross-sections. Then we have $\beta_1, ..., \beta_{k'} \in N(M)$ such that $g_i = \sum_{j=1}^{k'} \beta_j \alpha'_j$. It follows $\Phi = \sum_{j=1}^{k'} \beta_j \alpha_{j,1} \bmod \mathcal{M}_1$. Thus Φ is a global cross-section.

Apply the next theorem to the projection $\mathcal{N} \to \mathcal{N}/\mathcal{M}_1$ and Φ. Then there exists $g_{i,1} \in N(M)$ such that $g_{i,1x} = \Phi_x \bmod \mathcal{M}_{1x}$ for $x \in M$ and hence $(g_{i,1}, ..., g_{i,n}) \in H^0(M, \mathcal{M})$. $\square$

Theorem 2.8. *(Nash case of Cartan Theorem B) Let $h : \mathcal{M}_1 \to \mathcal{M}_2$ be a surjective $\mathcal{N}$-homomorphism between finite sheaves of $\mathcal{N}$-modules on a Nash manifold M. Assume that $\mathcal{M}_1$ is finitely generated by its global cross-sections. Then the induced map $h_* : H^0(M, \mathcal{M}_1) \to H^0(M, \mathcal{M}_2)$ is surjective.*

Proof. We can assume that $\mathcal{M}_1 = \mathcal{N}^n$ for some $n > 0 \in \mathbf{N}$ because there exist global generators $g_1, ..., g_n$ of $\mathcal{M}_1$ and then we can replace h with the surjective homomorphism $\mathcal{N}^n \supset \mathcal{N}_x^n \ni (\phi_1, ..., \phi_n) \to h(\sum_{i=1}^n \phi_i g_{ix}) \in \mathcal{M}_{2x} \subset \mathcal{M}_2,\ x \in M$. Set $\mathcal{M} = \operatorname{Ker} h$. Then by theorem 2.6, $\mathcal{M}$ is a finite sheaf of $\mathcal{N}$-submodules of $\mathcal{N}^n$, and $h : \mathcal{N}^n \to \mathcal{M}_2$ coincides with the projection $p : \mathcal{N}^n \to \mathcal{N}^n/\mathcal{M}$. Hence we consider p in place of h. Assume that $n > 1$ and the theorem holds for smaller n.

Let $f \in H^0(M, \mathcal{N}^n/\mathcal{M})$. We need to find $g \in H^0(M, \mathcal{N}^n) = N(M)^n$ such that $p_*(g) = f$. Let $\mathcal{M}_0$ denote the sheaf of $\mathcal{N}$-ideals with $\mathcal{M}_{0x} = \{0\}$ for $x \in M$. Then the homomorphism $\mathcal{M}_0 \times \mathcal{N}^{n-1} \to \mathcal{N}^n/(\mathcal{M} + \mathcal{N} \times \mathcal{M}_0 \times \cdots \times \mathcal{M}_0)$ is surjective and we can regard it as the projection $\mathcal{N}^{n-1} \to \mathcal{N}^{n-1}/\mathcal{L}$ for some finite sheaf of $\mathcal{N}$-submodules $\mathcal{L}$ of $\mathcal{N}^{n-1}$. Hence by induction hypothesis there exists $(0, g_2, ..., g_n) \in H^0(M, \mathcal{M}_0 \times \mathcal{N}^{n-1})$ whose image in $H^0(M, \mathcal{N}^n/(\mathcal{M} + \mathcal{N} \times \mathcal{M}_0 \times \cdots \times \mathcal{M}_0))$ coincides with the image of f there. Replace f with the difference of f and the image of $(0, g_2, ..., g_n)$ in $H^0(M, \mathcal{N}^n/\mathcal{M})$. Then we can assume from the beginning that $f \in H^0(M, (\mathcal{M} + \mathcal{N} \times \mathcal{M}_0 \times \cdots \times \mathcal{M}_0)/\mathcal{M})$. Hence we regard f as a global cross-section of $\mathcal{N} \times \mathcal{M}_0 \times \cdots \times \mathcal{M}_0/(\mathcal{M} \cap \mathcal{N} \times \mathcal{M}_0 \times \cdots \times \mathcal{M}_0)$ since $(\mathcal{M} + \mathcal{N} \times \mathcal{M}_0 \times \cdots \times \mathcal{M}_0)/\mathcal{M}$ is naturally isomorphic to $\mathcal{N} \times \mathcal{M}_0 \times \cdots \times \mathcal{M}_0/(\mathcal{M} \cap \mathcal{N} \times \mathcal{M}_0 \times \cdots \times \mathcal{M}_0)$. It was shown in the proof of theorem 2.6 that $\mathcal{M} \cap \mathcal{N} \times \mathcal{M}_0 \times \cdots \times \mathcal{M}_0$ is finite. Hence f is the image of some global cross-section g of $\mathcal{N} \times \mathcal{M}_0 \times \cdots \times \mathcal{M}_0$ under the projection $H^0(M, \mathcal{N} \times \mathcal{M}_0 \times \cdots \times \mathcal{M}_0) \to H^0(M, \mathcal{N} \times \mathcal{M}_0 \times \cdots \times \mathcal{M}_0/(\mathcal{M} \cap \mathcal{N} \times \mathcal{M}_0 \times \cdots \times \mathcal{M}_0))$ because this is the case of $\mathcal{M}_1 = \mathcal{N}$ in the theorem. Then $p_*(g) = f$. $\square$

Let X be a Nash subset of $\mathbf{R}^n$ and $f_1, ..., f_k$ be generators of the ideal of $N(\mathbf{R}^n)$ of functions vanishing on X. Let Sing X denote the subset of X where the Jacobian

matrix rank of $f_1, ..., f_k$ is smaller than $\operatorname{codim} X$. Let a complexification $X^{\mathbf{C}}$ of X in $\mathbf{C}^n$ be defined to be the common zero set of some complexifications $f_1^{\mathbf{C}}, ..., f_k^{\mathbf{C}}$ of $f_1, ..., f_k$. Then by Lemma 1.9 and Theorem 1.10 in [C-R-S$_2$] and theorem 2.7, we obtain the next remark.

Remark. $\operatorname{Sing} X$ is the smallest Nash subset of X whose complement is a Nash manifold. But it does not coincide in general with points in X where the germ of X is not a Nash manifold germ of $\dim X$. Moreover $\operatorname{Sing} X$ is also equal to $X \cap \operatorname{Sing} X^{\mathbf{C}}$, where $\operatorname{Sing} X^{\mathbf{C}}$ denotes the C^ω singular point set of $X^{\mathbf{C}}$.

We deduce from [Hi] a Nash version of Hironaka Desingularization Theorem that will be useful in our context.

Theorem 2.9. *(Nash case of Main Theorem I of [Hi]) Let X be a Nash subset of $\mathbf{R}^n$. Then there exists a finite sequence of blowings-up $X_r \xrightarrow{\pi_r} \cdots \xrightarrow{\pi_1} X_0 = X$ along smooth Nash centers $C_i \subset X_i$, $i = 1, ..., r-1$, such that X_r is smooth and $C_i \subset \operatorname{Sing} X_i$.*

Proof. Since $N(\mathbf{R}^n)$ is a Noetherian ring ([E] and [Ri]), we have generators $f_1, ..., f_k$ of the ideal of $N(\mathbf{R}^n)$ of functions vanishing on X. Set $F = (f_1, ..., f_k)$, which is a Nash map from $\mathbf{R}^n$ to $\mathbf{R}^k$, and $Y = \operatorname{graph} F$. Let Y^Z denote the Zariski closure of Y in $\mathbf{R}^n \times \mathbf{R}^k$ and let $\widetilde{Y^Z} \subset \mathbf{R}^n \times \mathbf{R}^k \times \mathbf{R}^{n'}$ be an algebraic set such that the restriction p to $\widetilde{Y^Z}$ of the projection $\mathbf{R}^n \times \mathbf{R}^k \times \mathbf{R}^{n'} \to \mathbf{R}^n \times \mathbf{R}^k$ is the normalization of Y^Z (we simply call $\widetilde{Y^Z}$ the *normalization* of Y^Z). Then by Artin-Mazur Theorem (see Theorem I.5.1 in [S$_2$]) there exists a connected component L of $\widetilde{Y^Z}$ consisting of only regular points such that $p(L) = Y$ and $p|_L : L \to Y$ is a Nash diffeomorphism. Let $q_1 : \widetilde{Y^Z} \to \mathbf{R}^n$ and $q_2 : \widetilde{Y^Z} \to \mathbf{R}^k$ denote the restrictions to $\widetilde{Y^Z}$ of the projections $\mathbf{R}^n \times \mathbf{R}^k \times \mathbf{R}^{n'} \to \mathbf{R}^n$ and $\mathbf{R}^n \times \mathbf{R}^k \times \mathbf{R}^{n'} \to \mathbf{R}^k$, respectively. Then $q_1|_L$ is a Nash diffeomorphism onto $\mathbf{R}^n$, the set $q_2^{-1}(0)$ is algebraic, the equality $(q_1|_L)^{-1}(X) = (q_2|_L)^{-1}(0)$ holds, and $(q_1|_L)^{-1}(\operatorname{Sing} X)$ is equal to the intersection of L with the algebraic singular point set of $q_2^{-1}(0)$. Indeed, $(q_1|_L)^{-1}(\operatorname{Sing} X)$ is contained in the above intersection because $(q_1|_L)^{-1}(\operatorname{Sing} X)$ is the smallest Nash subset of $(q_1|_L)^{-1}(X)\,(= (q_2|_L)^{-1}(0))$ whose complement is a Nash manifold (by the remark before theorem 2.9), and the converse inclusion follows from the equality $q_2 = F \circ q_1$ on L. Hence we can replace X by $L \cap q_2^{-1}(0)$—the union of some connected components of $q_2^{-1}(0)$. By Main Theorem I there exists a finite sequence of blowings-up $\tilde{X}_r \xrightarrow{\tilde{\pi}_r} \cdots \xrightarrow{\tilde{\pi}_1} \tilde{X}_0 = q_2^{-1}(0)$ along smooth algebraic centers $\tilde{C}_i \subset \tilde{X}_i$, for $i = 0, ..., r-1$, such that $\tilde{X}_r$ is smooth and $\tilde{C}_i \subset \operatorname{Sing} \tilde{X}_i$. Then $\tilde{X}_r \cap (\tilde{\pi}_1 \circ \cdots \circ \tilde{\pi}_r)^{-1}(L) \to \cdots \to \tilde{X}_0 \cap L$ fulfills the requirements. □

A sheaf of $\mathcal{N}$-($\mathcal{O}$-)ideals on a Nash (analytic) manifold M is called *normal crossing* if there exists a local Nash (analytic) coordinate system $(x_1, ..., x_n)$ of M at each point such that the stalk of the sheaf is generated by $\prod_{i=1}^n x_i^{\alpha_i}$ for some $(\alpha_1, ..., \alpha_n) \in \mathbf{N}^n$.

Theorem 2.10. *(Nash case of Main Theorem II of [Hi]) Let M be a Nash manifold and let $\mathcal{I}_1$ and $\mathcal{I}_2$ be finite sheaves of non-zero $\mathcal{N}$-ideals on M. Assume that $\mathcal{I}_2$ is normal crossing. Then there exists a finite sequence of blowings-up $M_r \xrightarrow{\pi_r} \cdots \xrightarrow{\pi_1} M_0 = M$ along smooth Nash centers $C_i \subset M_i$, for $i = 1, ..., r-1$, such that $(\pi_1 \circ \cdots \circ \pi_r)^{-1}\mathcal{I}_1\mathcal{I}_2\mathcal{N}_{M_r}$ is normal crossing, each C_i is normal crossing with*

$(\pi_1 \circ \cdots \circ \pi_i)^{-1}(\operatorname{supp} \mathcal{N}_M/\mathcal{I}_2) \cup \cup_{j=1}^{i} (\pi_j \circ \cdots \circ \pi_i)^{-1}(C_{j-1})$ *and* $\pi_1 \circ \cdots \circ \pi_i(C_i)$ *is contained in the subset of* M *consisting of* x *such that even* $\mathcal{I}_{1x}$ *is not generated by any power of one regular function germ or* $\mathcal{I}_{1x} + \mathcal{I}_{2x} \neq \mathcal{N}_x$.

Note that $(\pi_1 \circ \cdots \circ \pi_i)^{-1}\mathcal{I}_2\mathcal{N}_{M_i}$, for $i = 1, ..., r$, are normal crossing.

Proof. Let $f_1, ..., f_{k'} \in N(\mathbf{R}^n)$ and $f_{k'+1}, ..., f_k \in N(\mathbf{R}^n)$ be global generators of $\mathcal{I}_1$ and $\mathcal{I}_2$ (theorem 2.7), respectively, and define $F, Y, Y^Z, \widetilde{Y^Z}, L, q_1 : \widetilde{Y^Z} \to \mathbf{R}^n$ and $q_2 : \widetilde{Y^Z} \to \mathbf{R}^k$ as in the last proof. Let W be the subset of $\widetilde{Y^Z}$ consisting of points where $f_{k'+1} \circ q_1, ..., f_k \circ q_1$ do not generate a normal crossing sheaf of $\mathcal{N}$-ideals. Consider the algebraic $\mathbf{R}$-scheme of the topological underlying space $\widetilde{Y^Z} - \operatorname{Sing} \widetilde{Y^Z} - W$, and let $\mathcal{J}_1$ and $\mathcal{J}_2$ denote the sheaf of ideals of the scheme generated by $f_1 \circ q_1, ..., f_{k'} \circ q_1$ and by $f_{k'+1} \circ q_1, ..., f_k \circ q_1$, respectively. Then we can replace $M, \mathcal{I}_1$ and $\mathcal{I}_2$ with the scheme, $\mathcal{J}_1$ and $\mathcal{J}_2$. Hence the theorem follows from Main Theorem II. $\square$

Remark. Note that main Theorems I and II of [Hi] state some additional conditions that are automatically satisfied in the Nash case.

2.3. Topologies on function spaces.

Let M be a C^∞ manifold. We use three kinds of topologies on $C^\infty(M)$ as a topological linear space.

The first is the classical compact-open C^r topology, $r = 0, ..., \infty$, for which $C^\infty(M)$ is a Fréchet space if $r = \infty$.

The second is the *Whitney* C^r *topology*, $r = 0, ..., \infty$. Even if it is well-known, we recall its definition because we will define the third topology below by comparison with it. If $M = \mathbf{R}^n$, then a system of open neighborhoods of the zero function in $C^\infty(M)$ is given by

$$U_{r',g_\alpha} = \{f \in C^\infty(\mathbf{R}^n) : |D^\alpha f(x)| < g_\alpha(x),\, \alpha \in \mathbf{N}^n,\, |\alpha| \le r\}$$

where r' runs in $\{m \in \mathbf{N} : m \le r\}$ and g_α runs in $C^\infty(\mathbf{R}^n)$ with $g_\alpha > 0$ everywhere for each $\alpha \in \mathbf{N}^n$ with $|\alpha| \le r'$. If M is an open subset of $\mathbf{R}^n$, we define the topology on $C^\infty(M)$ in the same way. In general, embed M in some $\mathbf{R}^n$ and let $p : V \to M$ be the orthogonal projection of a tubular neighborhood of M in $\mathbf{R}^n$. Then p induces an injective linear map $C^\infty(M) \ni f \to f \circ p \in C^\infty(V)$ whose image is closed in $C^\infty(V)$ in the Whitney C^r topology. Hence $C^\infty(M)$ inherits a topology as a closed subspace of $C^\infty(V)$. We call it the Whitney C^∞ topology.

The *strong Whitney* C^∞ *topology* is the third topology which we will consider. Assume first that $M = \mathbf{R}^n$, and let g_α be a positive-valued C^∞ function on $\mathbf{R}^n$ and K_α be a compact subset of $\mathbf{R}^n$ for each $\alpha \in \mathbf{N}^n$ such that $\{\mathbf{R}^n - K_\alpha\}$ is locally finite. Set $g = (g_\alpha)_\alpha$ and $K = (K_\alpha)_\alpha$. Then a system of open neighborhoods of the zero function in $C^\infty(\mathbf{R}^n)$ is given by the family of sets

$$U_{g,\alpha} = \{f \in C^\infty(\mathbf{R}^n) : |D^\alpha f(x)| < g_\alpha(x) \text{ for } x \in \mathbf{R}^n - K_\alpha \text{ for } \alpha \in \mathbf{N}^n\}$$

for all g and K. We define the strong Whitney C^∞ topology on a general manifold M in the same way as in the case of the Whitney C^∞ topology. Moreover, we shall need to consider C^∞ functions on an analytic set and the strong Whitney C^∞ topology on the space. To this aim, we use another equivalent definition of the topology. Let $\{M_i\}$ be a family of compact C^∞ submanifolds of M possibly with

boundary such that $\{\operatorname{Int} M_l\}$ is a locally finite covering of M. Regard $C^\infty(M)$ as a subset of $\prod_l C^\infty(M_l)$ by the injective map $C^\infty(M) \ni f \to \prod_l f|_{M_l} \in \prod_l C^\infty(M_l)$. Then the family of sets $C^\infty(M) \cap \prod_l O_l$ is the system of open sets of $C^\infty(M)$, where O_l are open subsets of $C^\infty(M_l)$ in the C^∞ topology. Note that the product topology of $\prod_l C^\infty(M_l)$ induces the compact-open C^∞ topology.

For an analytic manifold M, we endow $C^\omega(M)$ with the three topologies in the same way, and we extend naturally the definition of the topologies to the spaces of C^∞ or C^ω maps between C^∞ or C^ω manifolds.

Remark 2.11. The first three remarks explain essential differences between the three topologies.

(1) The compact-open C^∞ topology, the Whitney C^∞ topology and the strong Whitney C^∞ topology coincide if M is compact.

(2) The strong Whitney C^∞ topology is stronger than the Whitney C^∞ topology if M is not compact.

(3) $C^\infty(M)$ is not a Fréchet space in the Whitney C^r topology nor the strong Whitney C^∞ topology if M is not compact. Indeed, it is even not metrizable.

The following remarks will be useful in the sequel.

(4) Whitney Approximation Theorem—any C^∞ function on an analytic manifold is approximated by a C^ω function—holds also in any of these topologies (see [W]).

Finally, an advantage of the strong Whitney C^∞ topology is that we can reduce many global problems to local problems using partition of unity.

(5) Let $\{\phi_i\}$ be a partition of unity of class C^∞ on M. Then for a neighborhood U of 0 in $C^\infty(M)$ in the strong Whitney C^∞ topology there exists another V such that if $f \in V$ then $\phi_i f \in U$ for all i and conversely if $\phi_i f \in V$ for all i then $f \in U$.

Let M be a Nash manifold. We give a topology on $N(M)$, called the *semialgebraic C^r topology*, $r = 0, ..., \infty$, so that a system of open neighborhoods of 0 in $N(M)$ is given by the family U_{r',g_α} defined in the above definition of the Whitney C^r topology, where g_α runs here in $N(M)$ only. If $r = \infty$, we call it the *Nash topology*. For $r < \infty$, let $N^r(M)$ denote the space of semialgebraic C^r functions on M. We define *semialgebraic $C^{r'}$ topology* on $N^r(M)$ for $r' \le r$ in the same way. We do not need the analog on $N(M)$ of the strong Whitney C^∞ topology. When M is not compact, it is the discrete topology by Proposition VI.2.8, [S$_2$] and next remark. A *partition of unity of class semialgebraic C^r*, $r \in \mathbf{N}$, on M is a **finite** family of non-negative semialgebraic C^r functions on M whose sum equals 1.

Remark 2.11,(5)$'$. Let $r' \le r \in \mathbf{N}$, and let $\{\phi_i\}$ be a partition of unity of class semialgebraic $C^{r'}$ on M. Then for a neighborhood U of 0 in $N^r(M)$ in the $C^{r'}$ topology there exists a neighborhood V of 0 in $N^r(M)$ such that if $f \in V$ then $\phi_i f \in U$ for all i and conversely if $\phi_i f \in V$ for all i then $f \in U$.

The reason is that $\{\phi_i\}$ is a finite family and the map $N^r(M) \ni f \to \phi_i f \in N^r(M)$ is continuous for each i by lemma II.1.6, [S$_2$], which states that $N^r(M)$ and $N(M)$ are topological rings in the semialgebraic $C^{r'}$ topology.

We need also the following lemma many times.

Lemma 2.12. *Let M be an analytic manifold and $\xi_1, ..., \xi_l$ be analytic functions on M. Then the maps $\Xi^\infty : C^\infty(M)^l \ni (h_1, ..., h_l) \to \sum_{i=1}^l \xi_i h_i \in \sum_{i=1}^l \xi_i C^\infty(M)$ and $\Xi^\omega : C^\omega(M)^l \ni (h_1, ..., h_l) \to \sum_{i=1}^l \xi_i h_i \in \sum_{i=1}^l \xi_i C^\omega(M)$ are open in both the compact-open C^∞ topology and the strong Whitney C^∞ topology.*

Note that in the case $l = 1$, the lemma is much easier to prove because the involved maps are injective. Moreover, the lemma does not necessarily hold in the Whitney C^∞ topology. This is one reason why we need to have recourse to the strong Whitney C^∞ topology in the paper.

Proof. Consider Ξ^∞ in the compact-open C^∞ topology. It is well-known that the ideal of $C^\infty(M)$ generated by a finite number of analytic functions is closed in $C^\infty(M)$ in any of the C^∞ topologies (which follows from Theorems III.4.9 and VI.1.1′, [Ml]). In particular $\sum_{i=1}^l \xi_i C^\infty(M)$ is a Fréchet space in the compact-open C^∞ topology, and Ξ^∞ is open by the open mapping theorem on Fréchet spaces. Note that the above proof is still valid in the case of an analytic manifold with corners.

Consider Ξ^∞ in the strong Whitney C^∞ topology. Let M_j be compact C^ω submanifolds of M with boundary such that $\{\operatorname{Int} M_j\}$ is a locally finite covering of M. Let $\{\phi_j\}$ be a partition of unity of class C^∞ subordinate to $\{\operatorname{Int} M_j\}$. As shown above, the map $C^\infty(M_j)^l \ni (h_1, ..., h_l) \to \sum_{i=1}^l \xi_i|_{M_j} h_i \in \sum_{i=1}^l \xi_i|_{M_j} C^\infty(M_j)$ is open for each j. Hence for each $h = (h_1, ..., h_l) \in C^\infty(M)^l$ and $g \in \sum_{i=1}^l \xi_i C^\infty(M)$ sufficiently close to $\sum_{i=1}^l \xi_i h_i$ in the strong Whitney C^∞ topology there exist $g_j = (g_{1,j}, ..., g_{l,j}) \in C^\infty(M_j)^l$ close to $h|_{M_j}$ such that $\sum_{i=1}^l \xi_i|_{M_j} g_{i,j} = g|_{M_j}$ for any j. Then $\sum_j \phi_j g_j$ is a well-defined C^∞ map from M to $\mathbf{R}^l$ and close to h by remark 2.11,(5), and $\sum_{i=1}^l \xi_i \sum_j \phi_j g_{i,j} = \sum_j \phi_j g = g$. Thus Ξ^∞ is also open in the strong Whitney C^∞ topology.

We finally consider Ξ^ω only in the strong Whitney C^∞ topology (the proof is similar, and even easier, in the compact-open C^∞ topology). Let $(h_1, ..., h_l) \in C^\omega(M)^l$ such that $\sum_{i=1}^l \xi_i h_i$ is small. Then, by openness of Ξ^∞, there exists small $(h'_1, ..., h'_l) \in C^\infty(M)^l$ such that $\sum_{i=1}^l \xi_i h'_i = \sum_{i=1}^l \xi_i h_i$ and hence $(h_1 - h'_1, ..., h_l - h'_l) \in \operatorname{Ker} \Xi^\infty$. Therefore, it suffices to see that $\operatorname{Ker} \Xi^\omega$ is dense in $\operatorname{Ker} \Xi^\infty$. Let $H = (h_1, ..., h_l) \in \operatorname{Ker} \Xi^\infty$. We want to approximate H by an element of $\operatorname{Ker} \Xi^\omega$.

Let $\mathcal{J}$ denote the kernel of the homomorphism : $\mathcal{O}^l \supset \mathcal{O}_a^l \ni (\phi_1, ..., \phi_l) \to \sum_{i=1}^l \xi_{ia} \phi_i \in \mathcal{O}_a \subset \mathcal{O}$, $a \in M$, which is a coherent sheaf of $\mathcal{O}$-submodules of $\mathcal{O}^l$ by theorem 2.5. Let $M^{\mathbf{C}}$ and $\mathcal{J}^{\mathbf{C}}$ be Stein and coherent complexifications of M and $\mathcal{J}$ which are complex conjugation preserving. Let $\{U_i\}$ be a locally finite open covering of $M^{\mathbf{C}}$ such that each $\overline{U}_i$ is compact. Let $H_{i,j} = (h_{1,i,j}, ..., h_{l,i,j})$, for $j = 1, ..., n_i$, $i = 1, 2, ...$, be global cross-sections of $\mathcal{J}$ such that $H_{i,j}|_M$ are real valued and $H_{i,1}, ..., H_{i,n_i}$ generate $\mathcal{J}^{\mathbf{C}}$ on U_i (theorem 2.1) for each i. Then $H_{i,1}|_{U_i \cap M}, ..., H_{i,n_i}|_{U_i \cap M}$ are generators of $\operatorname{Ker} \Xi^\infty|_{U_i \cap M}$. Actually, by Theorem VI,1.1′ in [Ml] it is equivalent to prove that $\operatorname{Ker} \Xi_a = \mathcal{F}_a \operatorname{Ker} \Xi_a^\omega$, $a \in U_i$, where $\mathcal{F}_a$ is the completion of $\mathcal{O}_a$ in the $\mathfrak{p}$-adic topology and the homomorphisms $\Xi_a^\omega : \mathcal{O}_a^l \to \mathcal{O}_a$ and $\Xi_a : \mathcal{F}_a^l \to \mathcal{F}_a$ are naturally defined. However, this condition is the flatness of $\mathcal{F}_a$ over $\mathcal{O}_a$, which is well-known (see [Ml]). Thus $H_{i,1}|_{U_i \cap M}, ..., H_{i,n_i}|_{U_i \cap M}$ generate $\operatorname{Ker} \Xi^\infty|_{U_i \cap M}$. Let $\{\rho_i\}$ be a partition of unity of class C^∞ subordinate to $\{U_i \cap M\}$. Then $\rho_i H \in \operatorname{Ker} \Xi^\infty|_{U_i \cap M}$ and we have C^∞ functions $\chi_{i,j}$ on M, for $j = 1, ..., n_i$, $i = 1, 2, ...$, such that $\operatorname{supp} \chi_{i,j} \subset U_i \cap M$ and $\rho_i H = \sum_{j=1}^{n_i} \chi_{i,j} H_{i,j}|_M$. By remark 2.11,(4) we can approximate $\chi_{i,j}$ by analytic functions $\chi'_{i,j}$. Moreover, as in [W] we can approximate so that each $\chi'_{i,j}$ can be complexified to a complex analytic function $\chi'^{\mathbf{C}}_{i,j}$ on $M^{\mathbf{C}}$ and $\sum_{i,j} |\chi'^{\mathbf{C}}_{i,j} H_{i,j}|$ is locally uniformly bounded. Then $\sum_{i,j} \chi'^{\mathbf{C}}_{i,j} H_{i,j}$ is a complex analytic map from $M^{\mathbf{C}}$ to $\mathbf{C}^l$, and its restriction to M

is both an approximation of H and an element of $\operatorname{Ker} \Xi^\omega$. Thus Ξ^ω is open. □

Lemma 2.12 holds in the Nash case for a compact Nash manifold. However, we do not know whether lemma 2.12 still holds for a non-compact Nash manifold. Consequently, we have recourse many times in this paper to compactification arguments that require much care to deal with.

3. Equivalence of normal crossing functions

3.1. On C^∞ equivalence of analytic functions with only normal crossing singularities.

Let us compare C^ω and C^∞ right equivalences of two analytic functions on an analytic manifold. The C^∞ right equivalence is easier to check. The C^ω right equivalence implies the latter. However the converse is not necessarily true. We will show that this is the case for analytic functions with only normal crossing singularities, and apply the fact to the proof of the main theorem 3.2.

The main theorem of this section is

Theorem 3.1. *(1) Let M be a C^ω manifold and $f, g \in C^\omega(M)$. Assume that f and g admit only normal crossing singularities. If f is C^∞ right equivalent to g, then f is C^ω right equivalent to g.*

(2) If C^∞ functions f and g on a C^∞ manifold M admit only normal crossing singularities and are proper and C^2 right equivalent, then they are C^∞ right equivalent.

(3) If f and g are semialgebraically C^2 right equivalent Nash functions on a Nash manifold M with only normal crossing singularities, then they are Nash right equivalent.

The case where M has corners also holds.

Remark. (i) The germ case is also of interest. Let M, f and g be the same as in above (1). Let ϕ be a C^∞ diffeomorphism of M such that $f = g \circ \phi$. Set $X = \operatorname{Sing} f$ and $Y = \operatorname{Sing} g$, and let $\{X_i\}$ and $\{Y_i\}$ be the irreducible analytic components of X and Y, respectively. Let A and B be the unions of some intersections of some X_i and Y_i, respectively. Assume that $\phi(A) = B$. Then we can choose a C^ω diffeomorphism π so that $f = g \circ \pi$ and $\pi(A) = B$. Consequently, theorem 3.1,(1) holds for the germs of f on A and g on B. Similar statements for (2) and (3) hold.

(ii) In the Nash case, C^∞ right equivalence does not imply Nash right equivalence. Indeed, let N be a compact contractible Nash manifold with non-simply connected boundary of dimension $n > 3$ (e.g., see [Mz]). Set $M = (\operatorname{Int} N) \times (0, 1)$ and let $f : M \to (0, 1)$ denote the projection. Then M and f are of class Nash, and M is Nash diffeomorphic to $\mathbf{R}^{n+1}$. Actually, smooth the corners of $N \times [0, 1]$. Then $N \times [0, 1]$ is a compact contractible Nash manifold with simply connected boundary of dimension strictly more than four. Hence by the positive answers to Poincaré conjecture and Schönflies problem (Brown-Mazur Theorem) $N \times [0, 1]$ is C^∞ diffeomorphic to an $(n + 1)$-ball. Hence by Theorem VI.2.2 in [S$_2$] M is Nash diffeomorphic to an open $(n + 1)$-ball. Let $g : M \to \mathbf{R}$ be a Nash function which is Nash right equivalent to the projection $\mathbf{R}^n \times (0, 1) \to (0, 1)$. Then f and g are C^ω right equivalent since $\operatorname{Int} N$ is C^ω diffeomorphic to $\mathbf{R}^n$, but they are not Nash equivalent because $\operatorname{Int} N$ and $\mathbf{R}^n$ are not Nash diffeomorphic, by Theorem VI 2.2 in [S$_2$].

For the proof of part (2) and (3) of the theorem, we need to prepare some material in next part. Therefore we postpone their proof to the last part of the paper.

Proof of theorem 3.1,(1). In this proof we apply the strong Whitney C^∞ topology unless otherwise specified. The idea of the proof is taken from [S$_1$]. Let us consider the case without corners. The proof is divided into three steps. Denote by X and Y the extended critical sets of f and g, that is $X = f^{-1}(f(\operatorname{Sing} f))$ and $Y = g^{-1}(g(\operatorname{Sing} g))$. Note that X and Y are not necessarily analytic sets. Let M be analytic and closed in the ambient Euclidean space $\mathbf{R}^N$, and consider the Riemannian metric on M induced from that of $\mathbf{R}^N$. Set $n = \dim M$.

Step 1. Assume that X is an analytic set. Let ϕ denote a C^∞ diffeomorphism of M such that $f = g \circ \phi$. Then there exists a C^ω diffeomorphism π of M arbitrarily close to ϕ such that $\pi(X) = Y$.

Proof of step 1. Let $\{X_i : i = 0, ..., n-1\}$ and $\{Y_i : i = 1, ..., n-1\}$ be the canonical stratifications of X and Y respectively, and put $X_n = M - X$ and $Y_n = M - Y$.

Before beginning the proof, we give some definitions and a remark. Fix X_i. Let $\{M_l\}$ be a family of compact C^∞ manifolds of dimension i possibly with boundary such that $\{\operatorname{Int} M_l\}$ is a locally finite covering of $\cup_{j=0}^i X_j$. A function on $\cup_{j=0}^i X_j$ is called *of class* C^∞ if its restriction to each M_l is of class C^∞. Thus $C^\infty(\cup_{j=0}^i X_j)$ is a subset of $\prod_l C^\infty(M_l)$. We give to $C^\infty(\cup_{j=0}^i X_j)$ the product topology of the C^∞ topology on each $C^\infty(M_l)$, i.e. the compact-open C^∞ topology. We give also the strong Whitney C^∞ topology on $C^\infty(\cup_{j=0}^i X_j)$ in the same way. Then lemma 2.12 holds for the map $C^\infty(\cup_{j=0}^i X_j) \ni f \to h|_{\cup_{j=0}^i X_j} f \in C^\infty(\cup_{j=0}^i X_j)$ for an analytic function h on M, which is proved in the same way. We will use this generalized version of the lemma below.

Let X' be a normal-crossing C^ω subset of M contained in X. Assume that the sheaf of $\mathcal{O}$-ideals on M defined by X' is generated by a single C^ω function ξ on M. Let V denote the subspace of $C^\infty(\cup_{j=0}^i X_j)$ consisting of functions which vanish on X'. Then $V = \xi C^\infty(\cup_{j=0}^i X_j)$ by Theorem VI,3.10 in [Ml]. We will use this remark later in this proof.

Now we begin the proof. By induction, for some $i \in \mathbf{N}$, assume that we have constructed a C^∞ diffeomorphism π_{i-1} of M close to ϕ such that $\pi_{i-1}|_{\cup_{j=0}^{i-1} X_j}$ is of class C^ω (in the sense that $\pi_{i-1}|_{\cup_{j=0}^{i-1} X_j} \in \prod_l C^\omega(M_l)$) and $\pi_{i-1}(X_j) = Y_j$ for any j. Let $\mathcal{M}$ denote the sheaf of $\mathcal{O}$-ideals on M defined by $\cup_{j=0}^{i-1} X_j$, which is coherent because X is normal-crossing. Then $\pi_{i-1}|_{\cup_{j=0}^{i-1} X_j} \in H^0(M, \mathcal{O}/\mathcal{M})^N$ for the following reason. As the problem is local, we can assume that $M = \mathbf{R}^n$ and $X = \{(x_1, ..., x_n) \in \mathbf{R}^n : x_1 \cdots x_{n'} = 0\}$ for some $n' \le n \in \mathbf{N}$. Moreover, we suppose that $i = n$ because if for each irreducible analytic component E of $\cup_{j=0}^i X_j$ we can extend $\pi_{i-1}|_{E \cap \cup_{j=0}^{i-1} X_j}$ to an analytic map on E then the extensions for all E define an analytic map from $\cup_{j=0}^i X_j$ to $\mathbf{R}^N$, and hence it suffices to work on each E in place of $\mathbf{R}^n$. Then what we see is that an analytic function on $X = \{(x_1, ..., x_n) \in \mathbf{R}^n : x_1 \cdots x_{n'} = 0\}$ is an element of $H^0(M, \mathcal{O}/\mathcal{M})$, namely, can be extensible to an analytic function on $\mathbf{R}^n$ (corollary 2.4), where $\mathcal{M}$ is defined by X. We proceed by induction on n'. Since the statement is clear if $n' = 0$ or $n' = 1$, assume that $n' > 1$ and that the restriction of f to $\{x_1 = 0\}$ is extensible to an analytic function $\tilde{f}$ on $\mathbf{R}^n$. Then $f - \tilde{f}|$ vanishes on $\{x_1 = 0\}$ and hence is

divisible by x_1. Apply the induction hypothesis to $(f-\tilde{f})/x_1$ and $\{x_2 \cdots x_{n'} = 0\}$ and let $\tilde{\tilde{f}}$ be an extension of $(f-\tilde{f})/x_1$. Then $\tilde{f} + x_1\tilde{\tilde{f}}$ is the required extension of f.

Consider any C^ω extension $\alpha : M \to \mathbf{R}^N$ of $\pi_{i-1}|_{\cup_{j=0}^{i-1} X_j}$ (corollary 2.4). Here we can choose α to be sufficiently close to π_{i-1} and so that $\operatorname{Im}\alpha \subset M$ for the following reason. Let $\gamma_1, ..., \gamma_k \in C^\omega(M)$ be generators of $\mathcal{M}$ (corollary 2.2). Then there exist $\delta_1, ..., \delta_k \in C^\infty(M, \mathbf{R}^N)$ such that $\pi_{i-1} - \alpha = \sum_{j=1}^k \gamma_j \delta_j$ by the above remark. Let $\tilde{\delta}_j$ be C^ω approximations of δ_j. Replace α with the composite of $\alpha + \sum \gamma_j \tilde{\delta}_j$ and the orthogonal projection of a neighborhood of M in $\mathbf{R}^N$ to M. Then α satisfies the requirements. Let $p_j : U_j \to Y_j$ be the orthogonal projection of a tubular neighborhood of Y_j in $\mathbf{R}^N$. Here U_j is described as $\cup_{y \in Y_j}\{x \in \mathbf{R}^N : \ |x-y| < \epsilon(y),\ (x-y) \perp T_yY_j\}$ for some positive C^0 function ϵ on Y_j where T_yY_j denotes the tangent space of Y_j at y, and we can choose ϵ so large that $\epsilon(y) \geq \epsilon_0 \operatorname{dis}(y, \cup_{k=0}^{j-1} Y_k)$ locally at each point of $\cup_{k=0}^{j-1} Y_k$ for some positive number ϵ_0 because Y is normal crossing. Then α can be so close to π_{i-1} that $\alpha(X_i) \subset U_i$ since $\alpha = \pi_{i-1}$ on $\cup_{j=0}^{i-1} X_j$ and hence $d\alpha_x v = d\pi_{i-1x} v$ for any $x \in \cup_{j=0}^{i-1} X_j$ and for any tangent vector v at x tangent to $\cup_{j=0}^{i-1} X_j$. Define π_i on $\cup_{j=0}^{i} X_j$ to be α on $\cup_{j=0}^{i-1} X_j$ and $p_i \circ \alpha$ on X_i. Note that π_i is a C^ω map from $\cup_{j=0}^{i} X_j$ to $\cup_{j=0}^{i} Y_j \subset \mathbf{R}^N$ and close to $\pi_{i-1}|_{\cup_{j=0}^{i} X_j}$. Actually $\pi_{i-1}|_{\cup_{j=0}^{i} X_j} = \overline{p_i} \circ \pi_{i-1}|_{\cup_{j=0}^{i} X_j}$ and moreover since α is close to π_{i-1}, then $\overline{p_i} \circ \alpha$ on $\cup_{j=0}^{i} X_j$ $(= \pi_i)$ is close to $\overline{p_i} \circ \pi_{i-1}$ on $\cup_{j=0}^{i} X_j$, where $\overline{p_i} : \overline{U_i} \to \cup_{j=0}^{i} Y_j$ is the natural extension of p_i. We need to extend π_i to a C^∞ diffeomorphism of M which is close to π_{i-1} and carries each X_j to Y_j. Compare $\pi_i \circ \pi_{i-1}^{-1}|_{\cup_{j=0}^{i} Y_j}$ and the identity map of $\cup_{j=0}^{i} Y_j$. Then they are close each other and what we have to prove is the following statement: let τ be a C^∞ map between $\cup_{j=0}^{i} Y_j$ close to id. Then we can extend τ to a C^∞ diffeomorphism of M which is close to id and carries each Y_j to Y_j, $j = i+1, ..., n$.

By the second induction, it suffices to extend τ to a C^∞ map between $\cup_{j=0}^{i+1} Y_j$ close to id. We reduce the problem to a trivial case. First it is enough to extend τ to a C^∞ map from $\cup_{j=0}^{i+1} Y_j$ to $\mathbf{R}^N$ close to id by virtue of $p_{i+1} : U_{i+1} \to Y_{i+1}$ as above. Secondly, if we replace τ with $\tau - \operatorname{id}|_{\cup_{j=0}^{i} Y_j}$ then the problem is that for a C^∞ map $\tau : \cup_{j=0}^{i} Y_j \to \mathbf{R}^N$ close to the zero map we can extend τ to a C^∞ map from $\cup_{j=0}^{i+1} Y_j$ to $\mathbf{R}^N$ close to 0. Thirdly, we can assume that τ is a function. Hence we can use a partition of unity of class C^∞ and the problem becomes local (remark 2.11,(5)). So we assume that $M = \mathbf{R}^n$, Y is the union of some irreducible analytic components of $\{y_1 \cdots y_n = 0\}$ and $\tau \in C^\infty(\cup_{j=0}^{i} Y_j)$ is close to 0 and vanishes on $\{y \in \cup_{j=0}^{i} Y_j \subset \mathbf{R}^n : \ |y| \geq 1\}$, where $(y_1, ..., y_n) \in \mathbf{R}^n$. Let ξ be a C^∞ function on M such that $\xi = 1$ on $\{y \in \mathbf{R}^n : \ |y| \leq 1\}$ and $\xi = 0$ on $\{|y| \geq 2\}$. If $n = 0$ or 1 we have nothing to do. Hence by the third induction on n, assume that we have a C^∞ extension $\tau_1(y_2, ..., y_n)$ of $\tau|_{\{y_1=0\} \cap \cup_{j=0}^{i} Y_j}$ to $\{y_1 = 0\} \cap \cup_{j=0}^{i+1} Y_j$ close to 0. Regard $\tau_1(y_2, ..., y_n)$ as a C^∞ function on $\cup_{j=0}^{i+1} Y_j$, which is possible because $\cup_{j=0}^{i+1} Y_j$ is contained in the product of $\mathbf{R}$ and the image of $\{y_1 = 0\} \cap \cup_{j=0}^{i+1} Y_j$ under the projection $\mathbf{R} \times \mathbf{R}^{n-1} \to \mathbf{R}^{n-1}$. Replace τ with $\tau - \tau_1\xi$, which vanishes on $\{y_1 = 0 \text{ or } |y| \geq 2\} \cap \cup_{j=0}^{i} Y_j$. Next consider $(\tau - \tau_1\xi)/y_1|_{\{y_2=0\} \cap \cup_{j=0}^{i} Y_j}$ and apply the generalized lemma 2.12, the above remark and the same arguments as above. Then we reduce the problem to the case where $\tau = 0$ on $\{y_1 y_2 = 0 \text{ or } |y| \geq 2\} \cap \cup_{j=0}^{i} Y_j$

and by the fourth induction to the case where $\tau = 0$ on $\cup_{j=0}^{i} Y_j$. Thus step 1 is proved.

Step 2. Assume that $X = Y$, X is an analytic set and there exists a C^∞ diffeomorphism ϕ of M such that $f = g \circ \phi$. Then there exists a C^ω diffeomorphism π of M close to ϕ such that $f = g \circ \pi$.

Proof of step 2. By step 1 there exists a C^ω diffeomorphism ϕ' of M arbitrarily close to ϕ such that $\phi'(X) = X$. Then $f \circ \phi'^{-1} = g \circ \phi \circ \phi'^{-1}$, $f \circ \phi'^{-1}$ is analytic and $\phi \circ \phi'^{-1}$ is close to id. Hence we assume in step 2 that g is fixed and f and ϕ can be chosen so that ϕ and $f - g$ are arbitrarily close to id and 0 respectively. We construct π by integrating along a well-chosen vector field on M. There exist analytic vector fields $w_1, ..., w_N$ on M which span the tangent space at each point of M, e.g., $w_{ix} = dp_x \frac{\partial}{\partial x_i}$, $x \in M$, where $(x_1, ..., x_N) \in \mathbf{R}^N$ and p is the orthogonal projection of a tubular neighborhood of M in $\mathbf{R}^N$. Consider a vector field $v = \frac{\partial}{\partial t} + \sum_{i=1}^{N} a_i w_i$ on $M \times [0, 1]$ where $a_i \in C^\omega(M \times [0, 1])$ for $i \in \{1, ..., N\}$. Put $F(x, t) = (1 - t) f(x) + t g(x)$ for $(x, t) \in M \times [0, 1]$.

Assume that we have found such a_i, $i = 1, ..., N$, that $v(F) = 0$ and $|\sum_{i=1}^{N} a_i w_i|$ is small. Then F is constant along integral curves of v, therefore, the flow of v furnishes an analytic diffeomorphism π so that $f = g \circ \pi$.

Therefore, what we have to do is to construct the relevant a_i, $i \in \{1, ..., N\}$. First look at the local case. We will show that there exist a compact neighborhood U of each point of M and $a_i \in C^\omega(U \times [0, 1])$, $i = 1, ..., N$, such that $v(F) = 0$ on $U \times [0, 1]$ and $|\sum_{i=1}^{N} a_i w_i|$ is small. If the point is in X, we can write $U = \{x \in \mathbf{R}^n : \ |x| \le 1\}$, $g(x) - c = \prod_{i=1}^{n} x_i^{n_i}$ and $f(x) - c = \lambda(x)(g(x) - c)$ for $x \in U$, where $c \in \mathbf{R}$, λ is a C^ω function on U and close to 1 by lemma 2.12, and at least one of n_i's, say n_1, is non-zero. Assume that $c = 0$ without loss of generality. Then there exists v of the form $\frac{\partial}{\partial t} + b_1 \frac{\partial}{\partial x_1}$, $b_1 \in C^\omega(U \times [0, 1])$, which satisfies $v(F) = 0$. Actually

$$\Big(\frac{\partial}{\partial t} + b_1 \frac{\partial}{\partial x_1}\Big) F(x, t) =$$

$$(1 - \lambda) g(x) + b_1(x, t)\big(n_1(t + (1 - t)\lambda(x))\frac{g(x)}{x_1} + (1 - t) g(x) \frac{\partial \lambda}{\partial x_1}(x)\big) = 0,$$

$$b_1(x, t) = \frac{-(1 - \lambda(x)) x_1}{n_1(t + (1 - t)\lambda(x)) + (1 - t) x_1 \frac{\partial \lambda}{\partial x_1}(x)},$$

which is an analytic function in $U \times [0, 1]$ and close to 0. Shrink U if necessary. Then for some $0 < i_1 < \cdots < i_n \le N$, the vector fields $w_{i_1}, ..., w_{i_n}$ span the tangent space there, and $b_1 \frac{\partial}{\partial x_1}$ is described uniquely by $\sum_{j=1}^{n} a_{i_j} w_{i_j}$ for some C^ω functions a_{i_j}. Hence a_{i_j}, $j = 1, ..., n$, and $a_i = 0$, $i \notin \{i_1, ..., i_n\}$, fulfill the requirements.

Next consider the situation at a point outside of X. Note that the values of f and g at the point may be different, and hence the above arguments do not work. We can choose its local coordinate system so that $U = \{x \in \mathbf{R}^n : \ |x| \le 1\}$ and $\frac{\partial g}{\partial x_1} = 1$ on U. Then

$$\Big(\frac{\partial}{\partial t} + b \frac{\partial}{\partial x_1}\Big) F(x, t) = -f + g + b_1\Big((1 - t)\frac{\partial f}{\partial x_1} + t \frac{\partial g}{\partial x_1}\Big) = 0,$$

$$b_1(x, t) = \frac{f - g}{(1 - t)\frac{\partial f}{\partial x_1} + t \frac{\partial g}{\partial x_1}},$$

and $(1-t)\frac{\partial f}{\partial x_1}+t\frac{\partial g}{\partial x_1}$ and $f-g$ are close to 1 and 0, respectively. Hence there exist U and a_i as before.

Consequently, using a partition of unity of class C^∞ we obtain a C^∞ vector field $v'=\frac{\partial}{\partial t}+\sum_{i=1}^N a_i'w_i$ on $M\times[0,\,1]$ such that $v(F)=0$ and $|\sum_{i=1}^N a_i'w_i|$ is small (remark 2.11,(5)).

Now, to construct the global analytic vector filed v on $M\times[0,\,1]$ we use Cartan Theorems A and B. Consider the sheaf of relations $\mathcal{J}$ on $M\times[0,\,1]$ defined by

$$\mathcal{J}=\cup_{(x,t)\in M\times[0,\,1]}\{(\beta,\alpha_1,\dots,\alpha_N)\in\mathcal{O}^{N+1}_{(x,t)}: \\ \beta(f_x-g_x)+\sum_{i=1}^N\alpha_i(w_i((1-t)f+tg))_{(x,t)}=0\}.$$

The sheaf $\mathcal{J}$ is a coherent sheaf of $\mathcal{O}$-modules by Oka Theorem 2.5. Later we will find $l\in\mathbf{N}$ and global cross-sections $(b_k,a_1^k,\dots,a_N^k)\in H^0(M\times[0,\,1],\mathcal{J})$, $k\in\{1,\dots,l\}$, such that for any $(x,t)\in M\times[0,\,1]$, any C^ω vector field germ ω at (x,t) in $M\times[0,\,1]$ with $\omega(F_{(x,t)})=0$ is of the form $\sum_{k=1}^l\xi_kv_{k(x,t)}$ for some C^ω function germs ξ_k at (x,t) in $M\times[0,1]$, where $v_k=b_k\frac{\partial}{\partial t}+\sum_{i=1}^N a_i^kw_i$. Assume the existence of such l and $(b_k,a_1^k,\dots,a_N^k)$. Then by the above method of construction of v' and by a partition of unity of class C^∞ there exist C^∞ functions θ_k on M such that $v'=\sum_{k=1}^l\theta_kv_k$. Approximate θ_k by C^ω functions $\tilde\theta_k$, and set $\tilde v=\sum_{k=1}^l\tilde\theta_kv_k$. Then $\tilde v$ is a C^ω vector field close to v' such that $F(\tilde v)=0$ and is described by $\tilde a_0\frac{\partial}{\partial t}+\sum_{i=1}^N\tilde a_iw_i,\ \tilde a_i\in C^\omega(M\times[0,\,1])$, for the following reason. Let $\mathcal{I}$ denote the coherent sheaf of $\mathcal{O}$-submodules of the sheaf of $\mathcal{O}$-modules of germs of C^ω vector fields on $M\times[0,\,1]$ defined by

$$\mathcal{I}_{(x,t)}=\{\omega:\ \omega(F_{(x,t)})=0\}\quad\text{for }(x,t)\in M\times[0,\,1],$$

and define an $\mathcal{O}$-homomorphism $\delta:\mathcal{O}^l\to\mathcal{I}$ by

$$\delta(\gamma_1,...,\gamma_l)=\sum_{k=1}^l\gamma_kv_{k(x,t)}\quad\text{for }(\gamma_1,...,\gamma_l)\in\mathcal{O}^l_{(x,t)},\ (x,t)\in M\times[0,\,1].$$

Then δ is surjective, $H^0(M\times[0,\,1],\mathcal{I})$ is the set of all C^ω vector fields w on $M\times[0,\,1]$ with $w(F)=0$, and hence by Cartan Theorem B the homomorphism $C^\omega(M\times[0,\,1])^l\ni(d_1,...,d_l)\to\sum_{k=1}^l d_kv_k\in H^0(M\times[0,\,1],\mathcal{I})$ is surjective, i.e., $\tilde v$ is of the form $\sum_{k=1}^l d_kv_i$ for some C^ω functions d_k on $M\times[0,\,1]$. Therefore, we have $\tilde v=\tilde a_0\frac{\partial}{\partial t}+\sum_{i=1}^N\tilde a_iw_i$ for $\tilde a_0=\sum_{k=1}^l d_k$ and $\tilde a_i=\sum_{k=1}^l d_ka_i^k$, $i=1,...,N$. Here $\tilde a_0$ is unique and hence close to 1, and $|\sum_{i=1}^N\tilde a_iw_i|$ is small. Thus $v=\frac{\partial}{\partial t}+\sum_{i=1}^N(\tilde a_i/\tilde a_0)w_i$ is what we wanted.

It remains to find $(b_k,a_1^k,...,a_N^k)$, $k=1,...,l$. That is equivalent to prove that $H^0(M\times[0,\,1],\mathcal{I})$ is finitely generated by Cartan Theorem B because the homomorphism $\mathcal{J}_{(x,t)}\ni(\beta,\alpha_1,...,\alpha_N)\to\beta\frac{\partial}{\partial t}+\sum_{i=1}^N\alpha_iw_{i(x,t)}\in\mathcal{I}_{(x,t)}$ is surjective. Moreover, it suffices to see that each stalk $\mathcal{I}_{(x,t)}$ is generated by a uniform number of elements by corollary 2.2. Note that F is an analytic function with only normal crossing singularities. Hence we replace F with f to simplify notation. Let $\mathcal{K}$ denote the sheaf of $\mathcal{O}$-modules of C^ω vector field germs on M such that

$$\mathcal{K}_x=\{\omega:\omega(f_x)=0\}\quad\text{for }x\in M.$$

Then it suffices to choose $l \in \mathbf{N}$ so that for any $x_0 \in M$, $\mathcal{K}_{x_0}$ is generated by l elements. Since the problem is local we can assume that $M = \mathbf{R}^n$, $x_0 = 0$ and $f(x) = \prod_{i=1}^k x_i^{n_i}$ with $n_1, ..., n_k > 0$, $0 < k \le n$. Write $\omega \in \mathcal{K}_0$ as $\sum_{i=1}^n \alpha_i \frac{\partial}{\partial x_i}$, $\alpha_i \in \mathcal{O}_0$, and set $h(x) = \prod_{i=1}^k x_i$. Then $\omega(f_{x_0}) = 0$ means

$$\sum_{i=1}^k n_i \alpha_i f(x)/x_i = 0, \quad \text{hence } \sum_{i=1}^k n_i \alpha_i h(x)/x_i = 0.$$

Therefore, each α_i is divisible by x_i. Hence, setting $\alpha_i' = \alpha_i/x_i$ we obtain $\sum_{i=1}^k n_i \alpha_i' = 0$. It is clear that $\{(\alpha_1', ..., \alpha_n') \in \mathcal{O}_0^n : \sum_{i=1}^k \alpha_i' = 0\}$ is generated by $n-1$ elements, which proves step 2.

The proof of step 1 shows that any C^∞ diffeomorphism of M carrying $\operatorname{Sing} f$ to $\operatorname{Sing} g$ is approximated by an analytic diffeomorphism of M with the same property. Hence it suffices to prove the next statement.

Step 3. Assume that $\operatorname{Sing} f = \operatorname{Sing} g$ and there exists a C^∞ diffeomorphism ϕ of M such that $f = g \circ \phi$. Then there exists a C^ω diffeomorphism π of M such that $f = g \circ \pi$.

Proof of step 3. As at the beginning of the proof of step 2 we fix g and modify f and ϕ so that ϕ and $f - g$ are sufficiently close to id and 0 respectively. Set $Z = \operatorname{Sing} f$ and let Z_i, $i = 1, 2, ...$, be connected components of Z. Let U_i be disjoint small open neighborhoods of Z_i in M such that if $\phi(U_i) \cap U_{i'} \ne \emptyset$ then $i = i'$. Then by steps 1 and 2 there exist C^ω diffeomorphisms $\pi_i : U_i \to \phi(U_i)$ close to $\phi|_{U_i} : U_i \to \phi(U_i)$ such that $f = g \circ \pi_i$ on U_i. Note that if we define a map between M to be π_i on each U_i and ϕ elsewhere, then the map is a C^∞ diffeomorphism by the definition of the strong Whitney C^∞ topology. For $x_0 \in M$, let $m(x_0)$ denote the multiplicity of $g - g(x_0)$ at x_0, i.e., $m(x_0) = |\alpha| = \alpha_1 + \cdots + \alpha_n$ for $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbf{N}^n$ such that $g(x) - g(x_0)$ is written as $\pm x^\alpha$ for some local coordinate system $(x_1, ..., x_n)$ at x_0. There exists $h \in C^\omega(M)$ such that $h^{-1}(0) = Z$ and h is $m(x)$-flat at each $x \in Z$ for the following reason. For each i, let $\{Z_{i,j}\}_j$ be the stratification of Z_i by multiplicity number, and for each $Z_{i,j}$, consider the smallest analytic set in U_i and hence in M containing each connected component of $Z_{i,j}$. Then we have a locally finite decomposition of Z_i into irreducible analytic sets $\{W_{i,j}\}_j$ in M such that $m(x)$ is constant, say $m_{i,j}$, on each $W_{i,j} - \cup_{j'}\{W_{i,j'} : \dim W_{i,j'} < \dim W_{i,j}\}$. By corollary 2.2 there exists $h_{i,j} \in C^\omega(M)$—e.g., the $m_{i,j}$th power of the square sum of a finite number of global generators of the sheaf of $\mathcal{O}$-ideals defined by $W_{i,j}$—such that $h_{i,j}^{-1}(0) = W_{i,j}$ and $h_{i,j}$ is $m_{i,j}$-flat at $W_{i,j}$, and then considering the sheaf of $\mathcal{O}$-ideals $\prod_{i,j} h_{i,j}\mathcal{O}$ we obtain h in the same way.

We will reduce the problem to the case where $\pi_i - \text{id}$ on U_i and $f - g$ are divisible by h. Since $\operatorname{supp} \mathcal{O}/h\mathcal{O} = Z$, $\{\pi_i\}_i$ defines an element of $H^0(M, (\mathcal{O}/h\mathcal{O})^N)$. Hence applying Cartan Theorem B to the exact sequence $0 \to (h\mathcal{O})^N \to \mathcal{O}^N \to (\mathcal{O}/h\mathcal{O})^N \to 0$, we obtain $\pi' \in C^\omega(M)^N$ such that $\pi_i - \pi' \in hC^\omega(U_i)^N$ for each i. We need to modify π' to be a diffeomorphism of M. Let ξ be a C^∞ function on M such that $\xi = 0$ outside of a small neighborhood of Z and $\xi = 1$ on a smaller one. Approximate C^∞ maps $\sum_i \xi(\pi_i - \pi')/h$ and $(1 - \xi)(\phi - \pi')/h$ from M to $\mathbf{R}^N$ by C^ω maps H_1 and H_2, respectively. Then $hH_1 + hH_2 + \pi'$ is an analytic approximation of $\phi : M \to \mathbf{R}^N$ whose difference with π_i on U_i is divisible by h. Hence its composite π'' with the orthogonal projection of M in $\mathbf{R}^N$ is an analytic

approximation of $\phi : M \to M$ such that $\pi_i - \pi''$ is divisible by h by the next fact. Given $\theta_1, \theta_2 \in \mathbf{R}\langle\langle x_1, ..., x_n\rangle\rangle^m$, $\eta \in \mathbf{R}\langle\langle x_1, ..., x_n\rangle\rangle$ and $\rho \in \mathbf{R}\langle\langle y_1, ..., y_m\rangle\rangle$ with $\theta_1(0) = \theta_2(0) = \eta(0) = \rho(0) = 0$, then $\rho \circ \theta_1 - \rho \circ (\theta_1 + \eta\theta_2)$ is divisible by η as an element of $\in \mathbf{R}\langle\langle x_1, ..., x_n\rangle\rangle$. Now replace ϕ, π_i and f with $\phi \circ \pi''^{-1}$, $\pi_i \circ \pi''^{-1}$ and $f \circ \pi''^{-1}$, respectively. Then the equalities $f = g \circ \phi$ and $f = g \circ \pi_i$ continue to hold, and $\pi_i - \mathrm{id}$ and hence $f - g$ are divisible by h and, moreover, by h^{3+s} by the same way, where $s \in \mathbf{N}$ is such that α^s is contained in the ideal of $\mathbf{R}\langle\langle x_1, ..., x_n\rangle\rangle$ generated by $\frac{\partial \psi_l}{\partial x_1}, ..., \frac{\partial \psi_l}{\partial x_n}$ for $\psi_l(x) = \prod_{j=1}^{l} x_j \in \mathbf{R}\langle\langle x_1, ..., x_n\rangle\rangle$, $l \le n$, and for $\alpha \in \mathbf{R}\langle\langle x_1, ..., x_n\rangle\rangle$ which vanishes on $\operatorname{Sing}\psi_l$ (Hilbert Zero Point Theorem). Set $h_1 = (f - g)/h^{3+s} \in C^\omega(M)$, which is close to 0 by lemma 2.12.

As in the proof of step 2, we define C^ω vector fields w_i, $i = 1, ..., N$, and a C^ω function F on $M \times [0, 1]$, and it suffices to find a C^ω vector field v of the form $\frac{\partial}{\partial t} + \sum_{i=1}^{N} a_i w_i$ on $M \times [0, 1]$ such that $v(F) = 0$ and $|\sum_{i=1}^{N} a_i w_i|$ is bounded. Since $f = g + h^{3+s}h_1$, then $F = g + (1 - t)h^{3+s}h_1$, and the equality $v(F) = 0$ becomes

$$h^{3+s}h_1 = \sum_{i=1}^{N} a_i(w_i g + (1 - t)h^{2+s}h_{2,i})$$

for some C^ω functions $h_{2,i}$ on M close to 0. This is solvable locally. Indeed, for each $x_0 \in M - Z$, at least one of $w_i g$, say $w_1 g$, does not vanish at x_0. Hence $a_1 = h^{3+s}h_1/(w_1 g + (1 - t)h^{2+s}h_{2,1})$, $a_2 = \cdots = a_N = 0$ is a solution on a neighborhood of x_0. Assume that $x_0 \in Z$. Then choose an analytic local coordinate system $(x_1, ..., x_n)$ at x_0 in M so that $g(x) = \prod_{i=1}^{n} x_i^{n_i} + \mathrm{const}$, where $\sum_{i=1}^{n} n_i = m(x_0) > 1$. Here we can assume that $x_0 = 0$, const $= 0$, $n_1, ..., n_l > 0$ and $n_{l+1} = \cdots = n_n = 0$. Note that $m(0, ..., 0, x_{l+1}, ..., x_n) = m(0)$ for $(x_{l+1}, ..., x_n) \in \mathbf{R}^{n-l}$ near 0. What we prove is that for each $t_0 \in [0, 1]$, the ideal I of $\mathcal{O}_{(0,t_0)}$ generated by $\frac{\partial g_{(0,t_0)}}{\partial x_i} + (1 - t_{(0,t_0)})h_{(0,t_0)}^{2+s}h_{2,i(0,t_0)}$, $i = 1, ..., l$, contains $h_{(0,t_0)}^{3+s}h_{1(0,t_0)}$. Let J denote the ideal of $\mathcal{O}_{(0,t_0)}$ generated by $\frac{\partial g_{(0,t_0)}}{\partial x_i}$, $i = 1, ..., l$. Then it suffices to see that $h_{(0,t_0)}^{1+s} \in J$ because if so, $J \supset I$, $J \ni h_{(0,t_0)}^{3+s}h_{1(0,t_0)}$, $J = I + \mathfrak{m}J$ and hence by Nakayama lemma $J = I$, where $\mathfrak{m}$ is the maximal ideal of $\mathcal{O}_{(0,t_0)}$. Moreover, assuming $g(x) = x_1 \cdots x_l$ we prove that $h_{(0,t_0)}^{s} \in J$, which is sufficient because $\frac{\partial g_{(0,t_0)}}{\partial x_i} = n_i \prod_{j=1}^{l} x_j^{n_j-1} \frac{\partial x_1 \cdots x_l}{\partial x_i}$ and $h_{(0,t_0)}$ is divisible by $\prod_{j=1}^{l} x_j^{n_j-1}$ by the definition of h. However, $h_{(0,t_0)}^{s} \in J$ is clear by the definition of s. Note that we can choose local $v = \frac{\partial}{\partial t} + \sum_{i=1}^{N} a_i w_i$ in any case so that $|\sum_{i=1}^{N} a_i w_i|$ is arbitrarily small.

We continue to proceed in the same way as in the proof of step 2. We obtain C^ω vector fields $v_k = b_k \frac{\partial}{\partial t} + \sum_{i=1}^{N} a_i^k w_i$, $k = 1, ..., l$, by local existence and a C^∞ vector field $v' = \frac{\partial}{\partial t} + \sum_{i=1}^{N} a_i' w_i$ such that $v_k(F) = v'(F) = 0$, $|\sum_{i=1}^{N} a_i' w_i|$ is small and v' is of the form $\sum_{k=1}^{l} \theta_k w_i$. After then we approximate θ_k by C^ω functions $\tilde{\theta}_k$, and $v = \sum_{k=1}^{l} \tilde{\theta} v_k$ fulfills the requirements. Thus we complete the proof of (1) in the case of without corners.

The case with corners is proved in the same way. $\square$

3.2. Cardinality of the set of equivalence classes.

Our main theorem establishes the cardinality of analytic (respectively Nash) R-L equivalence classes of analytic (respectively Nash) functions on M with only normal crossing singularities.

Theorem 3.2. *Let M be a compact analytic (respectively, Nash) manifold of strictly positive dimension. Then the cardinality of analytic (resp., Nash) R-L equivalence classes of analytic (resp. Nash) functions on M with only normal crossing singularities is 0 or countable. In the Nash case, the compactness of M is not necessary, and if moreover M is non-compact then the cardinality is countable.*

The proof of theorem 3.2 runs as follows. We reduce the C^ω case to the Nash case by proposition 4.1 and then the non-compact Nash case to the compact Nash case by proposition 4.9. Lemmas 4.3 and 4.4 together with Nash Approximation Theorem II prove the compact Nash case. We postpone the proof of theorem 3.2 to the last part of the paper.

Remark. (i) The case where the cardinality is zero may appear, e.g. $M = S^2, \mathbf{P}(2)$ (for the proof, see the arguments in (v) below in case $M = \mathbf{R}^2$).

(ii) In the theorem we do not need to fix M, namely, the cardinality of equivalence classes of analytic or Nash functions on all compact analytic manifolds or Nash manifolds, respectively, with only normal crossing singularities is also countable. Indeed, the cardinality is clearly infinite, and there are only a countable number of compact analytic manifolds and (not necessarily compact) Nash manifolds up to analytic diffeomorphism and Nash diffeomorphism, respectively, which will be clear in the proof of lemma 4.4.

(iii) Theorem 3.2 does not hold for analytic functions on a non-compact analytic manifold. To be precise, for a non-compact analytic manifold M, the cardinality of analytic R-L equivalence classes of (proper) analytic functions on M with only normal crossing singularities is of the continuum (0 or of the continuum, respectively). We prove this fact below.

(iv) On any non-compact connected analytic (Nash, respectively,) manifold M, there exists a non-singular analytic (Nash, respectively,) function. We give the construction below.

(v) An example of non-compact M where there is no proper analytic (Nash) function with only normal crossing singularities is $\mathbf{R}^2$. We see this by reduction to absurdity. Assume that there exists such an f. Note that each level of f is a finite union of Jordan curves. Let $a_1 \in \mathbf{R}$ be a point of $\operatorname{Im} f$ and $X_1 \subset \mathbf{R}^2$ be a Jordan curve in $f^{-1}(a_1)$ that does not intersect with $f^{-1}(a_1)$ inside of U_1. Next choose $a_2 \in f(U_1)$, a Jordan curve X_2 in $f^{-1}(a_2) \cap U_1$ and U_2 in the same way. If we continue these arguments, we arrive at a contradiction to the above note.

Proof of (iii) for proper functions. Assume that there exists a proper analytic function f on a non-compact analytic manifold M with only normal crossing singularities. Replacing f with $\pi \circ f$ for some proper analytic function π on $\mathbf{R}$ if necessary, we can assume that $f(\operatorname{Sing} f) = \mathbf{N}$ because $f(\operatorname{Sing} f)$ has no accumulating points in $\mathbf{R}$. Define a map $\alpha_f : \mathbf{N} \to \mathbf{N}$ so that for each $n \in \mathbf{N}$, $f - n$ is $\alpha_f(n)$-flat at any point of $f^{-1}(n) \cap \operatorname{Sing} f$ and not $(\alpha_f(n)+1)$-flat at some point of $f^{-1}(n) \cap \operatorname{Sing} f$. If a proper analytic function g with $g(\operatorname{Sing} g) = \mathbf{N}$ is C^ω R-L equivalent to f then $\alpha_f = \alpha_g$. Consider all proper C^ω functions π on $\mathbf{R}$ such that $\operatorname{Sing} \pi = \mathbf{N}$ and $\pi = \mathrm{id}$ on $\mathbf{N}$. Then the cardinality of $\{\alpha_{\pi \circ f}\}$ is of the continuum. Hence the cardinality of C^ω R-L equivalence classes of proper analytic functions on M with only normal crossing singularities is of the continuum. $\square$

Proof of (iv). Assume that $\dim M > 1$. We use the idea of handle body decomposition by Morse functions (see [Mi]). Let f be a non-negative proper C^∞ function on a non-compact connected C^ω manifold M with singularities of Morse

type. Approximate f and changing $\mathbf{R}$ by some C^ω diffeomorphism of $\mathbf{R}$, we assume that f is of class C^ω, that $f|_{\operatorname{Sing} f}$ is injective and $f(\operatorname{Sing} f) = 2\mathbf{N}$. For each $k \in \mathbf{N}$, let A_k be the union of $f^{-1}(k) \cap \operatorname{Sing} f$ with one point in each connected component of $f^{-1}(k)$ not containing points of $\operatorname{Sing} f$. Consider the 1-dimensional simplicial complex K whose 0-skeleton K^0 is $\cup_{k\in\mathbf{N}} A_k$ and whose 1-skeleton K^1 consists of 1-simplexes $\overline{ab}$, for $a, b \in K^0$, such that $f(\{a, b\}) = \{k, k+1\}$ for some $k \in \mathbf{N}$ and there exists a connected component C of $f^{-1}((k,\, k+1))$ with $\overline{C} \ni a, b$. Note that such a C is unique because $f_{f^{-1}((2k',\, 2k'+2))} : f^{-1}((2k',\, 2k'+2)) \to (2k',\, 2k'+2)$ is a proper submersion for $k' \in \mathbf{N}$ and that conversely for each connected component C of $f^{-1}((k,\, k+1))$ there exist $a, b \in K^0$ such that $f(\{a, b\}) = \{k, k+1\}$ and $\overline{C} \ni a, b$. In other words, we can identify K^1 with the set of connected components of $f^{-1}((k,\, k+1)) : k \in \mathbf{N}$. Moreover, for $\overline{ab} \in K^1$ there exist an injective C^ω map $l_{a,b} : [0,\, 1] \to M$ with $l_{a,b}(0) = a$, $l_{a,b}(1) = b$, $f \circ l_{a,b}(t) = f(a) \pm t$ and $\operatorname{Im} l_{a,b} = \operatorname{Im} l_{b,a}$. Here for $\overline{ab} \neq \overline{a'b'}$, then $\operatorname{Im} l_{a,b} \cap \operatorname{Im} l_{a',b'} = \{a\}$ if $a = a'$ or $a = b'$, or $\operatorname{Im} l_{a,b} \cap \operatorname{Im} l_{a',b'} = \{b\}$ if $b = a'$ or $b = b'$, and $\operatorname{Im} l_{a,b} \cap \operatorname{Im} l_{a',b'} = \emptyset$ otherwise. Hence we can identify the underlying polyhedron $|K|$ with the subset $\cup_{\overline{ab}\in K^1} \operatorname{Im} l_{a,b}$ of M, i.e., K is realized in M. Note also that there exists a unique C^0 retraction $r : M \to \cup_{\overline{ab}\in K^1} \operatorname{Im} l_{a,b}$ such that $f \circ r = f$.

We will see that each $a \in K^0$ is the end of some half-polygon in $|K|$, i.e., there exist distinct $a_0 = a, a_1, a_2, \ldots \in K^0$ such that $\overline{a_i a_{i+1}} \in K^1$ for $i \in \mathbf{N}$. Note that $a_i \to \infty$ (i.e., $f(a_i) \to \infty$) as $i \to \infty$. Since M is non-compact and connected, there exists a proper C^1 map $l : [0,\, \infty) \to M$ such that $l(0) = a$. We can move $\operatorname{Im} l$ into $|K|$ by r so that $\operatorname{Im} l$ is the underlying polyhedron of some subcomplex of K. If there is a 1-simplex s in $K_l \overset{\text{def}}{=} K|_{\operatorname{Im} l}$ with an end v not equal to $l(0)$ nor equal to another 1-simplex in K_l, then remove s and v from K_l, and repeat this operation as many times as possible. Then K_l becomes a simplicial subcomplex of K and $|K_l|$ is the union of a half-polygon and Jordan curves. Remove, moreover, some vertices except $l(0)$ and 1-simplexes so that $|K_l|$ is a half-polygon. Then we obtain an injective simplicial map $l : \tilde{\mathbf{N}} \to K$ with $l(0) = a$, where $\tilde{\mathbf{N}} = \mathbf{N} \cup \{[i,\, i+1] : i \in \mathbf{N}\}$. Let L_a denote all of such l, and let l_a be such that $\min f \circ l_a = \max\{\min f \circ l : l \in L_a\}$ and $\#(f \circ l_a)^{-1}(\min f \circ l_a) \le \#(f \circ l)^{-1}(\min f \circ l)$ for $l \in L_a$ with $\min f \circ l = \min f \circ l_a$.

Next we show that $\min f \circ l_a \to \infty$ as $a \to \infty$. Otherwise, there would exist distinct $a_1, a_2, \ldots$ in K^0 such that $\min f \circ l_{a_i}$ remains constant, say equal to m. Note that $a_i \to \infty$ as $i \to \infty$. Since $f^{-1}(m)$ is compact we have a subsequence of $a_1, a_2, \ldots$ where $\operatorname{Im} l_{a_i}$ contain one point $b_0 \in K^0$ with $f(b_0) = m$. Next, choose a subsequence so that $\operatorname{Im} l_{a_i}$ contain $\overline{b_0 b_1} \in K^0$ for some $b_1 \in K^0$ and $l_{a_i}(k_i + 1) = b_0$ and $l_{a_i}(k_i) = b_1$ for some $k_i \in \mathbf{N}$. Repeating these arguments we obtain sequences $a_1, a_2, \ldots$ and $b_0, b_1, \ldots$ in K^0 such that $l_{a_i}(k_i + i) = b_0, \ldots, l_{a_i}(k_i) = b_i$ for some $k_i \in \mathbf{N}$, $i = 1, 2, \ldots$ Then $\cup_{i\in\mathbf{N}} \overline{b_i b_{i+1}}$ is a half-polygon. Fix i so large that $f(b_j) > m$, $j = i, i+1, \ldots$, and consider a polyhedron $l_{a_i}([0,\, k_i]) \cup \overline{b_i b_{i+1}} \cup \overline{b_{i+1} b_{i+2}} \cup \cdots$. Remove vertices and open 1-simplexes from it, as in above construction of l, so that the polyhedron becomes a half-polygon starting from a_i. This half-polygon defines a new $l \in L_{a_i}$. Clearly $\min f \circ l \ge m = \min f \circ l_{a_i}$ for this l by the definition of l. However, $\min f \circ l = \min f \circ l_{a_i}$ by the definition of l_a. Then the difference between this l and l_{a_i} is $\#(f \circ l)^{-1}(m) < \#(f \circ l_{a_i})^{-1}(m)$ since $f \circ l_{a_i}(k_i + 1) = m$, the inclusion $f \circ l_{a_i}([0,\, k_i]) \supset f \circ l([0,\, k_i])$ holds and since $f(\overline{b_i b_{i+1}} \cup \overline{b_{i+1} b_{i+2}} \cup \cdots) > m$, which contradicts the definition of l_{a_i}. Thus $\min f \circ l_a \to \infty$ as $a \to \infty$.

Note that M is C^∞ diffeomorphic to $M - \operatorname{Im} l_a$ for any $a \in K^0$ and the diffeo-

morphism can be chosen to be id outside of a small neighborhood of $\operatorname{Im} l_a$. Hence if $\operatorname{Im} l_a \cap \operatorname{Im} l_{a'} = \emptyset$ for any $a \neq a' \in \operatorname{Sing} f$, there exists a C^ω diffeomorphism $\pi : M \to M - \cup_{a \in \operatorname{Sing} f} \operatorname{Im} l_a$ such that $f \circ \pi$ is a non-singular analytic function on M. Consider the case where $\operatorname{Im} l_a \cap \operatorname{Im} l_{a'} \neq \emptyset$ for some $a \neq a' \in \operatorname{Sing} f$. Set $\{a_0, a_1, ...\} = \operatorname{Sing} f$, set $X_0 = Y_0 = \operatorname{Im} l_{a_0}$ and $Z_0 = \emptyset$. Let $i \in \mathbf{N}$. Assume by induction that we have defined subpolyhedra $X_i \supset Y_i \supset Z_i$ of $|K|$. If $X_i \cap \operatorname{Im} l_{a_{i+1}} = \emptyset$, set $X_{i+1} = X_i \cup \operatorname{Im} l_{a_{i+1}}$, set $Y_{i+1} = \operatorname{Im} l_{a_{i+1}}$ and $Z_{i+1} = \emptyset$. Otherwise, set $X_{i+1} = X_i \cup \operatorname{Im} l_{a_{i+1}}([0, k_{i+1}])$, define Z_{i+1} to be the closure of the unbounded connected component of the set of difference of the connected component of X_i containing $l_{a_{i+1}}(k_{i+1})$) and of $l_{a_{i+1}}(k_{i+1})$, and set $Y_{i+1} = Z_{i+1} \cup \operatorname{Im} l_{a_{i+1}}([0, k_{i+1}])$, where $k_{i+1} = \min\{k \in \mathbf{N} : X_i \cap l_{a_{i+1}}([0, k]) \neq \emptyset\}$. Then $X = \cup_{i \in \mathbf{N}} X_i$ is the underlying polyhedron of a subcomplex of K, and for each i there exists a C^∞ diffeomorphism $\pi_i : M - Z_i \to M - Y_i$ such that $\pi_i = \mathrm{id}$ on $X_i - Z_i$ and outside of a small neighborhood of $Y_i - Z_i$ in $M - Z_i$. Since $\min f \circ l_a \to \infty$ as $a \to \infty$, we see that $\cdots \circ \pi_1 \circ \pi_0 : M \to M$ is a well-defined C^∞ diffeomorphism to $M - X$. Approximate it by a C^ω diffeomorphism $\pi : M \to M - X$. Then $f \circ \pi$ is the required non-singular analytic function on M.

Consider the case where M is a non-compact connected Nash manifold. Then there exists a proper Nash function on M with only singularities of Morse type. Actually, by Theorem VI.2.1 in [S_2], the manifold M is Nash diffeomorphic to the interior of a compact Nash manifold with boundary M', which is called a *compactification* of M. Then by using a partition of unity of class semialgebraic C^2 we obtain a nonnegative semialgebraic C^2 function ϕ on M' with zero set $\partial M'$ and with only singularities of Morse type. Approximating the semialgebraic C^2 function $1/\phi$ on M by a Nash function ψ in the semialgebraic C^2 topology (Approximation Theorem I), we obtain the required function. Note that $\# \operatorname{Sing} \psi < \infty$ because $\operatorname{Sing} \psi$ is semialgebraic. Hence in the same way as in the analytic case, we can find a Nash function on M without singularities by the following fact.

Let X be a 1-dimensional closed semialgebraic connected subset of M which is a union of smooth curves $X_0, ..., X_k$ such that any X_i is closed in M, any X_i and X_j intersect transversally and for each $a \in X$ there exists one and only one path from a to ∞ in X. Then M and $M - X$ are Nash diffeomorphic.

We prove this fact as follows. Assume that $M = \operatorname{Int} M'$ for M' as above. Then the closure $\overline{X}$ of X in M' intersects with $\partial M'$ at one point. By moving X by a semialgebraic C^1 diffeomorphism of M and then by a Nash diffeomorphism (Approximation Theorem I) we assume that $\overline{X}$ is smooth at $\overline{X} \cap \partial M'$ and $\overline{X}$ and $\partial M'$ intersect transversally. Let ξ denote the function on M' which measures distance from $\overline{X}$. This function being semialgebraic, we approximate $\xi|_{M' - \overline{X}}$ by a positive Nash function $\tilde{\xi}$ on $M' - \overline{X}$ so that $\tilde{\xi}(x) \to 0$ as x converges to a point of $\overline{X}$. Let $\epsilon > 0$ be small enough. Then $\tilde{\xi}|_{\tilde{\xi}^{-1}((0, \epsilon])} : \tilde{\xi}^{-1}((0, \epsilon]) \to (0, \epsilon]$ is a proper trivial Nash submersion by [Ha]. Hence $M' - \overline{X} - \tilde{\xi}^{-1}((0, \epsilon])$ and $M' - \overline{X}$ are semialgebraically C^1 diffeomorphic and, moreover, Nash diffeomorphic by Approximation Theorem I. On the other hand, $M' - \overline{X} - \tilde{\xi}^{-1}((0, \epsilon))$ is a compact Nash manifold with corners, and if we smooth the corners then $M' - \overline{X} - \tilde{\xi}^{-1}((0, \epsilon))$ is C^∞ and hence Nash (Theorem VI.2.2 in [S_2]) diffeomorphic to M' by the assumptions on X, which implies that $M - X - \tilde{\xi}^{-1}((0, \epsilon])$ and M are Nash diffeomorphic. Therefore, $M - X = \operatorname{Int}(M' - \overline{X})$ is Nash diffeomorphic to M. $\square$

Proof of (iii) for non-proper functions. As in the situation above, we have a non-

bounded non-singular non-negative analytic function f on a non-compact connected analytic manifold M. Let π be a proper analytic function on $\mathbf{R}$ such that $\operatorname{Sing}\pi = \mathbf{N}$ and $\pi = \mathrm{id}$ on $\mathbf{N}$. Then $\pi \circ f(\operatorname{Sing}\pi \circ f) = \mathbf{N}$. Hence as in the case of proper functions, we see that the cardinality of C^ω R-L equivalence classes of analytic functions on M with only normal crossing singularities is of the continuum. $\square$

4. Reductions

In order to prove theorems 3.1,(2) and 3.1,(3) and theorem 3.2, we proceed to some reductions. Firstly, we reduce the analytic case to the Nash one, secondly we reduce the non-compact Nash case to the compact one.

4.1. Reduction to the Nash case.

By the following proposition we reduce the C^ω case of theorem 3.2 to the Nash case.

Proposition 4.1. *Let M be a compact Nash manifold possibly with corners, and f a C^ω function on M with only normal crossing singularities. Then f is C^ω right equivalent to some Nash function.*

Remark. If M is a non-compact Nash manifold, proposition 4.1 does not hold. For example, consider $M = \mathbf{R}$ and $f(x) = \sin x$.

Proof of proposition 4.1. Set $X = f^{-1}(f(\operatorname{Sing} f))$. Let $g : \tilde{X} \to M$ be a C^ω immersion of a compact C^ω manifold possibly with corners such that $\operatorname{Im} g = X$, $g|_{g^{-1}(\operatorname{Reg} X)}$ is injective and $g_x(\tilde{X}_x)$ is an analytic subset germ of $M_{g(x)}$ for each $x \in \tilde{X}$. Here we construct $\tilde{X}$ and g locally and then paste them. For a connected component C of $\tilde{X}$ there are two possible cases to consider: either $g(C) \subset \partial M$ or $g(C) \not\subset \partial M$. Assume that $g(C) \not\subset \partial M$ for any C. Then $g(\operatorname{Int}\tilde{X}) \subset \operatorname{Int} M$ and $g(\partial\tilde{X}) \subset \partial M$, and moreover X is a normal crossing analytic subset of M. Consider the family of all C^∞ maps $g' : \tilde{X} \to M$ with $g'(\operatorname{Int}\tilde{X}) \subset \operatorname{Int} M$ and $g'(\partial\tilde{X}) \subset \partial M$. Then

Lemma 4.2. *Let $r\,(>0) \in \mathbf{N} \cup \{\infty\}$. Then g is C^∞ stable in family, in the sense that any such C^∞ map $g' : \tilde{X} \to M$ close to g in the C^r topology is C^∞ R-L equivalent to g.*

Remark. The proof we produce below shows that lemma 4.2 holds even if M is a non-compact Nash (C^∞) manifold possibly with corners, using the Whitney C^r (strong Whitney C^∞, respectively) topology.

Proof of lemma 4.2. It suffices to find a C^∞ diffeomorphism of M which carries $\operatorname{Im} g$ to $\operatorname{Im} g'$. As usual, using a tubular neighborhood of M in its ambient Euclidean space, the orthogonal projection to M and a partition of unity of class C^∞, we reduce the problem to the following local problem.

Assume that $M = \mathbf{R}^n \times [0,\infty)^m$ and $X = \{x_1 \cdots x_l = 0\}$, for $l \le n$. Let $y_1 = y_1(x)$ be a C^∞ function on M which is close to the function x_1 in the Whitney C^r topology and coincides with x_1 outside of a neighborhood of 0. Then there exists a C^∞ diffeomorphism π of M which is id outside of a neighborhood of 0 and close to id in the Whitney C^r topology and carries $\{x_2 \cdots x_l = 0\} \cup \{y_1(x) = 0\}$ to X.

This is true since $\pi(x_1, ..., x_{n+m}) = (y_1(x), x_2, ..., x_{n+m})$ satisfies the requirements. $\square$

Continued proof of proposition 4.1. Let $0 \ll r' \ll r \in \mathbf{N}$.

Case without corners. Give a Nash manifold structure to $\tilde{X}$ (Theorem of Nash, see Theorem I.3.6 in [S$_2$]). Let $g' : \tilde{X} \to M$ be a Nash approximation of g in the C^r topology, e.g., the composite of a polynomial approximation of the map g from $\tilde{X}$ to the ambient Euclidean space of M with the orthogonal projection of a tubular neighborhood of the space, and set $X' = \operatorname{Im} g'$. Then by lemma 4.2 there exists a C^∞ diffeomorphism π of M which carries X to X', and by the above proof π can be arbitrarily close to the identity map in the C^r topology. Let $t_1, ..., t_l$ be the critical values of f. We assume that $t_i > 0$. We want to construct a Nash function f' on M such that $(f')^{-1}(f'(\operatorname{Sing} f')) = X'$ and $f' \circ \pi = f$ on X for some modified π and, moreover, $f' \circ \pi - t_i$ and $f - t_i$ have the same multiplicity at each point of $f^{-1}(t_i)$ for each i. For each t_i, let $\mathcal{I}_i$ denote the sheaf of $\mathcal{N}$-ideals with zero set $\pi(f^{-1}(t_i))$ and having the same multiplicity as $f \circ \pi^{-1} - t_i$ at each point of $\pi(f^{-1}(t_i))$. Such a sheaf exists because a non-singular semialgebraic and analytic set germ is a non-singular Nash set germ. Then $\mathcal{I}_i$ is generated by a finite number of global Nash functions (theorem 2.7). Let ϕ_i denote the square sum of the generators and define a Nash function ψ_i on M so that $\psi_i^2 = \phi_i$ and ψ_i has the same sign as $f \circ \pi^{-1} - t_i$ everywhere. Note that $\psi_i^{-1}(0) = \pi(f^{-1}(t_i))$ and ψ_i and $f \circ \pi^{-1} - t_i$ have the same multiplicity at each point of $\psi_i^{-1}(0)$. Set $\phi = \prod \phi_i$. We have a global cross-section of the sheaf of $\mathcal{N}$-modules $\mathcal{N}/\prod_i \mathcal{I}_i^2$ whose value at each point x of $\psi_i^{-1}(0)$ equals $\psi_{ix} + t_i \mod \mathcal{I}_{ix}^2$. Apply theorem 2.8 to the homomorphism $\mathcal{N} \to \mathcal{N}/\prod_i \mathcal{I}_i^2$ and the global cross-section. Then there exists a Nash function ψ on M such that $\psi - t_i$ and $f \circ \pi^{-1} - t_i$ have the same sign at each point of a neighborhood of $\psi_i^{-1}(0)$ and the same multiplicity at each point of $\psi_i^{-1}(0)$ for each i. We need to modify ψ so that $X' = \psi^{-1}(\psi(\operatorname{Sing} \psi))$. Let f'' be a C^∞ function on M, constructed by a partition of unity of class C^∞, such that $f'' = \psi$ on a small neighborhood of X' and $X' = (f'')^{-1}(f''(\operatorname{Sing} f''))$. Then $f'' - \psi$ is of the form $\phi\xi$ for some C^∞ function ξ on M. Let $\tilde{\xi}$ be a strong Nash approximation of ξ in the C^∞ topology, and set $f' = \psi + \phi\tilde{\xi}$. Then f' is a Nash function, $X' = (f')^{-1}(f'(\operatorname{Sing} f'))$ and $f' - t_i$ and $f \circ \pi^{-1} - t_i$ have the same multiplicity at each point of $\pi(f^{-1}(t_i))$ for each i.

By Theorem 3.1,(1) it suffices to see that f is C^∞ right equivalent to the function h defined to be $f' \circ \pi$. Note that $h^{-1}(h(\operatorname{Sing} h)) = X$, and $f - t_i$ and $h - t_i$ have the same multiplicity at each point of $f^{-1}(t_i)$ for each i. Remember that π is close to id in the C^r topology. We can choose f' so that f and h are close each other in the C^r topology. Indeed, $f \circ \pi^{-1} - f'$ is of the form $\eta \prod_i \psi_i$ for some C^∞ function η on M. Replace f' with $f' + \tilde{\eta} \prod_i \psi_i$ for a strong Nash approximation $\tilde{\eta}$ of η in the C^∞ topology. Then f and h are close. Hence we can reduce the problem, as usual, to the following local problem.

Let $M = \mathbf{R}^n$, $f(x) = x_1^{\alpha_1} \cdots x_l^{\alpha_l}$ and $h(x) = a(x) x_1^{\alpha_1} \cdots x_l^{\alpha_l}$ for some C^∞ function $a(x)$ on M close to 1 in the Whitney $C^{r'}$ topology (lemma 2.12). Assume that $\alpha_1 > 0$. Then f and h are C^∞ right equivalent through a C^∞ diffeomorphism close to id in the Whitney $C^{r'}$ topology.

This is true since the C^∞ diffeomorphism $\mathbf{R}^n \ni (x_1, ..., x_n) \to (a^{1/\alpha_1}(x) x_1, x_2, ..., x_n) \in \mathbf{R}^n$ satisfies the requirements. Thus the case without corners is proved.

Case with corners. Let M_1 be a Nash manifold extension of M. We can assume that M is the closure of the union of some connected components of $M_1 - Y$ for a normal crossing Nash subset Y of M_1. Let U be an open semialgebraic neighborhood of M in M_1 so small that f is extensible to an analytic function f

on U with only normal crossing singularities. Shrinking U if necessary, we replace X in the above proof with $X_1 = f_1^{-1}(f_1(\operatorname{Sing} f_1))$, and we define a C^ω manifold $\tilde{X}_1$ and a C^ω immersion $g_1 : \tilde{X}_1 \to U$ in the same way. For each connected component C of $\tilde{X}_1$ there are two possible cases to consider: either $g_1(C) \subset Y$ or $g_1(C) \not\subset Y$. If $g_1(C) \subset Y$, then $g_1(C)$ is a Nash subset of U with only normal crossing singularities and C has an abstract Nash manifold structure such that $g_1|_C$ is a Nash diffeomorphism to $g_1(C)$ (see [S_2] for the definition of an abstract Nash manifold). Apply Artin-Mazur Theorem to $g_1(C)$. Then C with this abstract Nash manifold structure is a Nash manifold. Set $g_1' = g_1$ on C. If $g_1(C) \not\subset Y$, give a Nash manifold structure to C, approximate $g_1|_C$ by a Nash immersion $g_1'|_C : C \to M_1$. In this way we define a Nash immersion $g_1' : \tilde{X}_1 \to M_1$ and set $X' = \operatorname{Im} g_1' \cap M$. The rest proceeds in the same way as the case without corners. □

The following lemma is the C^ω or Nash version of lemma 4.2 and is used to prove theorems 3.1,(2), 3.1,(3) and lemma 4.4.

Lemma 4.3. *Let $r\,(> 0) \in \mathbf{N} \cup \{\infty\}$. Let M and N be compact C^ω manifolds possibly with corners such that $\dim M = 1 + \dim N$. Let $\phi : N \to M$ be a C^ω immersion such that $\phi(\operatorname{Int} N) \subset \operatorname{Int} M$, $\phi(\partial N) \subset \partial M$, $\operatorname{Im}\phi$ is a normal crossing analytic subset of M and the restriction of ϕ to $\phi^{-1}(\operatorname{Reg}\operatorname{Im}\phi)$ is injective. Then ϕ is C^ω stable in the family of C^ω maps from N to M carrying ∂N to ∂M in the same sense as in lemma 4.2. If M, N and ϕ are of class Nash, then ϕ is Nash stable in the family of Nash maps with the same property as above.*

Remark. In the case of a non-compact M and proper ϕ, we see easily that the former half part of lemma 4.3 holds in the Whitney C^r topology, $r\,(> 0) \in \mathbf{N} \cup \{\infty\}$. We can prove the latter half in the non-compact case in the semialgebraic C^r topology by reducing to the compact case by lemmas 4.5 and 4.6.

Proof of lemma 4.3. Let ψ be an analytic approximation of ϕ in family in the analytic case. Then by lemma 4.2 ψ is C^∞ R-L equivalent to ϕ, namely, there exists a C^∞ diffeomorphism π of M which carries $\operatorname{Im}\phi$ to $\operatorname{Im}\psi$. Note that we can choose π to be close to id in the C^r topology by the proof of lemma 4.2. Then by step 1 in the proof of theorem 3.1,(1) and its proof, we can choose an analytic π even in the case with corners. The existence of an analytic diffeomorphism τ of N with $\psi \circ \tau = \pi \circ \phi$ is clear because $\tau = \psi^{-1} \circ \pi \circ \phi$ on $\phi^{-1}(\operatorname{Reg}\operatorname{Im}\phi)$. Thus ϕ and ψ are C^ω R-L equivalent.

Assume that M, N, ϕ and ψ are of class Nash. It suffices to find a Nash diffeomorphism of M which carries $\operatorname{Im}\phi$ to $\operatorname{Im}\psi$. Let π be such a diffeomorphism of M of class C^ω.

Case without corners. Let $\mathcal{I}_\phi$ and $\mathcal{I}_\psi$ denote the sheaves of $\mathcal{N}$-ideals on M defined by $\operatorname{Im}\phi$ and $\operatorname{Im}\psi$, respectively, and let $\{f_i\}$ and $\{g_j\}$ be a finite number of their respective global generators. Then $\{g_j \circ \pi\}$ is a set of global generators of the sheaf of $\mathcal{O}$-ideals $\mathcal{I}_\phi\mathcal{O}$ on M. Hence in the same way as in step 2 of the proof of theorem 3.1,(1) we obtain C^ω functions $\alpha_{i,j}$ and $\beta_{i,j}$ on M such that

$$f_i = \sum_j \alpha_{i,j} \cdot (g_j \circ \pi) \quad \text{and} \quad g_j \circ \pi = \sum_i \beta_{i,j} f_i.$$

Let $M \subset \mathbf{R}^n$ and let h be a Nash function on $\mathbf{R}^n$ with zero set M. Extend g_j to Nash functions on $\mathbf{R}^n$ and use the same notation g_j (theorem 2.8). Consider the following

equations of Nash functions in variables $(x, y, a_{i,j}, b_{i,j}) \in M \times \mathbf{R}^n \times \mathbf{R}^{n'} \times \mathbf{R}^{n'}$, for some n':

$$h(y) = 0, \quad f_i(x) - \sum_j a_{i,j} g_j(y) = 0, \quad g_j(y) - \sum_i b_{i,j} f_i(x) = 0.$$

We have a C^ω solution $y = \pi(x)$, $a_{i,j} = \alpha_{i,j}(x)$ and $b_{i,j} = \beta_{i,j}(x)$. Hence by Nash Approximation Theorem II there exists a Nash solution $y = \pi'(x)$, $a_{i,j} = \alpha'_{i,j}(x)$ and $b_{i,j} = \beta'_{i,j}(x)$, which are close to π, $\alpha_{i,j}$ and $\beta_{i,j}$, respectively. Then

$$\operatorname{Im} \pi' = M, \quad f_i = \sum_j \alpha'_{i,j} \cdot (g_j \circ \pi'), \quad g_j \circ \pi' = \sum_i \beta_{i,j} f_i.$$

Hence π' is a Nash diffeomorphism of M and carries $\operatorname{Im} \phi$ to $\operatorname{Im} \psi$.

Case with corners. We can assume that for Nash manifold extensions M_1 and N_1 of M and N, respectively, ϕ, ψ are extensible to Nash immersion ϕ_1 and ψ_1 of N_1 into M_1 and π to a C^ω embedding π_1 of a semialgebraic open neighborhood U of M in M_1 into M_1, and moreover there exist normal crossing Nash subsets Y of M_1 and Z of N_1 such that M and N are closures of the unions of some connected components of $M_1 - Y$ and of $N_1 - Z$, respectively, $\phi_1(Z) \subset Y$ and $\psi_1(Z) \subset Y$. Let M_1 be contained and closed in an open semialgebraic set O in $\mathbf{R}^n$, and h_1 be a Nash function on O with zero set M_1. Take a small open semialgebraic neighborhood V of M in M_1 and shrink M_1, N_1 and U so that $\pi(U) \subset V$ and $U \cap \operatorname{Im} \phi_1$ and $V \cap \operatorname{Im} \psi_1$ are normal crossing Nash subsets of U and V, respectively. Then in the same way as above, we obtain a finite number of global generators $\{f_{1,i}\}$ and $\{g_{1,i}\}$ of the sheaves of $\mathcal{N}$-ideals on U and V defined by $U \cap \operatorname{Im} \phi_1$ and $V \cap \operatorname{Im} \psi_1$, respectively, and analytic functions $\alpha_{1,i,j}, \beta_{1,i,j}$ on U such that

$$f_{1,i} = \sum_j \alpha_{1,i,j} \cdot (g_{1,j} \circ \pi_1) \quad \text{and} \quad g_{1,j} \circ \pi_1 = \sum_i \beta_{1,i,j} f_{1.i} \quad \text{on } U.$$

We need to describe the condition $\pi(\partial M) = \partial M$, i.e., $\pi_1(U \cap Y) \subset Y$. Let ξ' be the square sum of a finite number of global generators of the sheaf of $\mathcal{N}$-ideals $\mathcal{I}$ on M_1 defined by Y. Then ξ' is a generator of $\mathcal{I}^2$, and since M is a manifold with corners there exists a unique Nash function ξ on a semialgebraic neighborhood of M in M_1 such that $\xi^2 = \xi'$ and $\xi > 0$ on $\operatorname{Int} M$. Shrink U once more. Then $\xi|_U$ and $\xi \circ \pi_1|_U$ are well-defined generators of $\mathcal{I}|_U$ and $\mathcal{I}\mathcal{O}|_U$, respectively, and we have a positive analytic function γ on U such that $\xi \circ \pi = \gamma \xi$ on U. We shrink O, and using the same notation we extend $g_{1,j}$ and ξ to Nash functions on O.

Consider the germs on $M \times O \times \mathbf{R}^{n'} \times \mathbf{R}^{n'} \times \mathbf{R}$ of the following equations of Nash functions in the variables $(x, y, a_{i,j}, b_{i,j}, c) \in U \times O \times \mathbf{R}^{n'} \times \mathbf{R}^{n'} \times \mathbf{R}$ for some n':

$$h_1(y) = 0, \ f_{1,i}(x) - \sum_j a_{i,j} g_{1,j}(y) = 0, \ g_{1,j}(y) - \sum_i b_{i,j} f_{1,i}(x) = 0, \ \xi(y) - c\xi(x) = 0.$$

Then, since Nash Approximation Theorem II holds in the case of germs, we have Nash germ solutions on M of the equations $y = \pi'_1(x)$, $a_{i,j} = \alpha'_{i,j}(x)$, $b_{i,j} = \beta'_{i,j}(x)$ and $c = \gamma'(x)$. The equation $\xi \circ \pi'_1 = \gamma' \xi$ means $\pi'_1(M) = M$. Thus $\pi'_1|_M$ is the required Nash diffeomorphism of M. $\square$

The following lemma shows countable cardinality of the normal crossing Nash (C^ω) subsets of a compact Nash (C^ω, respectively) manifold possibly with corners.

Lemma 4.4. *Let M be a compact Nash manifold possibly with corners of strictly positive dimension. Consider Nash immersions ϕ from compact Nash manifolds possibly with corners of dimension equal to $\dim M-1$ into M such that $\operatorname{Im}\phi$ are normal crossing Nash subsets of M, the restrictions $\phi|_{\phi^{-1}(\operatorname{Reg}\operatorname{Im}\phi)}$ are injective and ϕ carry the interior and the corners into the interior and the corners, respectively. Then the cardinality of Nash R-L equivalence classes of all the ϕ's is countable.*

The analytic case also holds.

Proof of lemma 4.4. Note that the cardinality is infinite because for any $k\in\mathbf{N}$ we can embed k copies of a sphere of dimension $\dim M-1$ in M. It suffices to treat only the Nash case for the following reason.

Let $\phi: M'\to M$ be an analytic immersion as in lemma 4.4 for analytic M' and M. Assume that M has no corners. Since a compact analytic manifold is C^ω diffeomorphic to a Nash manifold, we suppose that M' and M are Nash manifolds. Approximate ϕ by a Nash map. Then ϕ is C^ω R-L equivalent to the approximation by lemma 4.3. Hence we can replace ϕ by a Nash map.

Assume that M has corners. Let $M_1\subset\mathbf{R}^n$ be an analytic manifold extension of M such that M is the closure of the union of some connected components of M_1-Y for a normal crossing analytic subset Y of M_1. We can assume that M_1 is compact as follows. Let α denote the function on M_1 which measures distance from M. Approximate $\alpha|_{M_1-M}$ by a C^ω function α' in the Whitney C^∞ topology, and let $\epsilon<\epsilon'$ be positive numbers so small that $M\cup\alpha'^{-1}((0,\,\epsilon'])$ is compact and such that the restrictions of α' to $\alpha'^{-1}((\epsilon,\,\epsilon'))$ and to its intersections with strata of the canonical stratification of Y are regular. Then $(M_1\cap\alpha'^{-1}((\epsilon,\,\epsilon')),Y\cap\alpha'^{-1}((\epsilon,\,\epsilon')))$ is C^ω diffeomorphic to $(M_1\cap\alpha'^{-1}((\epsilon+\epsilon')/2)),Y\cap\alpha'^{-1}((\epsilon+\epsilon')/2)))\times(\epsilon,\,\epsilon')$. Hence, replacing M_1 with the double of $M\cup\alpha'^{-1}((0,\,(\epsilon+\epsilon')/2])$, we assume that M_1 is compact.

Next we reduce the problem to the case where M_1 and Y are of class Nash. Define, as in the proof of proposition 4.1, a C^ω immersion $g:\tilde{Y}\to M_1$ of a compact C^ω manifold so that $\operatorname{Im} g=Y$, so that $g|_{g^{-1}(\operatorname{Reg}\operatorname{Im} g)}$ is injective and $g_y(\tilde{Y}_y)$ is an analytic subset germ of $M_{1g(y)}$ for each $y\in\tilde{Y}$. Give Nash structures on M_1 and $\tilde{Y}$, and approximate g by a Nash map g'. Then by lemma 4.3 there exists a C^ω diffeomorphism π of M_1 which carries $\operatorname{Im} g$ to $\operatorname{Im} g'$. Hence we can replace Y with $\operatorname{Im} g'$ and we assume from the beginning that M_1, Y and M are of class Nash. By the same reason, we suppose that M' is a Nash manifold possibly with corners and the closure of the union of some connected components of M'_1-Y' for a compact Nash manifold extension M'_1 of M' and a normal crossing Nash subset Y' of M'_1. Extend ϕ to a C^ω immersion ϕ_1 of a compact semialgebraic neighborhood U of M' in M'_1 into M_1, choose U so small that $\phi_1(U\cap Y')\subset Y$, and approximate, as in the proof of step 1 in theorem 3.1,(1), ϕ_1 by a Nash map $\tilde{\phi}_1$ so that $\tilde{\phi}_1(U\cap Y')\subset Y$ (here we use theorems 2.7 and 2.8 in place of corollaries 2.2 and 2.4 in the proof in theorem 3.1,(1)). Then $\tilde{\phi}_1|_{M'}$ is a Nash immersion into M, and $\operatorname{Im}\tilde{\phi}_1|_{M'}$ is a normal crossing Nash subset of M, moreover $\tilde{\phi}_1|_{(\tilde{\phi}_1|_{M'})^{-1}(\operatorname{Reg}\operatorname{Im}\tilde{\phi}_1|_{M'})}$ is injective and $\tilde{\phi}_1(\partial M')\subset\partial M$, and finally $\tilde{\phi}_1|_{M'}$ is C^ω R-L equivalent to ϕ by lemma 4.3. Thus we reduce the analytic case to the Nash one.

Consider the Nash case. Let $M\subset\mathbf{R}^n$ and $\phi: M'\to M$ be a Nash immersion as in lemma 4.4. Let M_1, M'_1, Y and $\phi_1: M'_1\to M_1$ be Nash manifold extensions of M

such that $M_1 \subset \mathbf{R}^n$, $\phi_1 = \phi$ on M', M and M' are the closures of the unions of some connected components of $M_1 - Y$ and $M_1' - \phi_1^{-1}(Y)$, respectively, $U \cap \operatorname{Im}\phi_1$ is a normal crossing Nash subset of an open semialgebraic neighborhood U of M in M_1 and $\phi_1|_{\phi_1^{-1}(\operatorname{Reg}(U\cap \operatorname{Im}\phi_1))}$ is injective. By Artin-Mazur Theorem (see the proof of theorem 2.9) we can regard M_1' as an open semialgebraic subset of the regular point set of an algebraic variety in $\mathbf{R}^n \times \mathbf{R}^{n'}$ for some n' and ϕ_1 as the restriction to M_1' of the projection $\mathbf{R}^n \times \mathbf{R}^{n'} \to \mathbf{R}^n$. We will describe all such $\phi_1 : M_1' \to M_1$ with fixed complexity as follows. Any algebraic set in $\mathbf{R}^n \times \mathbf{R}^{n'}$, and its subset of regular points where the projection to $\mathbf{R}^n$ is regular, are, respectively, described by the common zero set of polynomial functions $f_1, ..., f_l$ on $\mathbf{R}^n \times \mathbf{R}^{n'}$ for some $l \in \mathbf{N}$ and

$$\bigcup_{\substack{I=\{i_1,...,i_k\}\subset\{1,...,l\}\\ I'=\{i'_1,...,i'_{k+n'}\}\subset\{1,...,l\}}} \{x = (x_1, ..., x_{n+n'}) \in \mathbf{R}^{n+n'} :$$

$$f_1(x) = \cdots = f_l(x) = 0, \quad |\frac{\partial(f_{i'_1}, ..., f_{i'_{k+n'}})}{\partial(x_{i_1}, ..., x_{i_k}, x_{n+1}, ..., x_{n+n'})}(x)| \neq 0,$$

$$g_{I',i''} f_{i''} = \sum_{j=1}^{k+n'} g_{I',i'',j} f_{i'_j}, \quad g_{I',i''}(x) \neq 0, \quad i'' \in \{1, ..., l\} - I'\}$$

for some polynomial functions $g_{I',i''}$ and $g_{I',i'',j}$ on $\mathbf{R}^n \times \mathbf{R}^{n'}$, where $k = n + 1 - \dim M$, and $\frac{\partial(\ )}{\partial(\ )}$ denotes the Jacobian matrix. Moreover, its open semialgebraic subset is its intersection with

$$\cup_{j'=1}^{l} \cap_{i=1}^{l} \{x \in \mathbf{R}^{n+n'} : h_{i,j'}(x) > 0\}$$

for some polynomial functions $h_{i,j'}$ on $\mathbf{R}^n \times \mathbf{R}^{n'}$ (here we enlarge l if necessary). Thus $\phi_1 : M_1' \to \mathbf{R}^n$ is described by the family f_i, $g_{I',i''}$, $g_{I',i'',j}$ and $h_{i,j'}$ and conversely, any polynomial functions f_i, $g_{I',i''}$, $g_{I',i'',j}$ and $h_{i,j'}$ define in the above way a Nash manifold M_1' in $\mathbf{R}^{n+n'}$ of dimension $\dim M - 1$ such that the projection $\phi_1 : M_1' \to \mathbf{R}^n$ is an immersion. If the degree of the polynomials are less than or equal to $d \in \mathbf{N}$, we say that $\phi_1 : M_1' \to \mathbf{R}^n$ is of complexity l, d, n'.

Furthermore, since a polynomial function on $\mathbf{R}^{n+n'}$ of degree less than or equal to d is of the form $\sum_{\alpha\in\mathbf{N}_d^{n+n'}} a_\alpha x^\alpha$, $a_\alpha \in \mathbf{R}$, where $\mathbf{N}_d^{n+n'} = \{\alpha \in \mathbf{N}^{n+n'} :\ |\alpha| \leq d\}$, we regard the family of $f_i, .., h_{i,j'}$ of degree less than or equal to d as an element $a = (a_\alpha)$ of $\mathbf{R}^N$ for some $N \in \mathbf{N}$. We write $\phi_1 : M_1' \to \mathbf{R}^n$ as $\phi_{1a} : M_{1a}' \to \mathbf{R}^n$. Then the set $X = \cup_{a\in\mathbf{R}^N}\{a\} \times M_{1a}' \subset \mathbf{R}^N \times \mathbf{R}^n \times \mathbf{R}^{n'}$ is semialgebraic, and we can identify $\phi_{1a} : M_{1a}' \to \mathbf{R}^n$ with $p|_{(q\circ p)^{-1}(a)} : (q \circ p)^{-1}(a) \to \{a\} \times \mathbf{R}^n$, where $p : X \to \mathbf{R}^N \times \mathbf{R}^n$ and $q : \mathbf{R}^N \times \mathbf{R}^n \to \mathbf{R}^N$ are the projections.

Consider the condition $\operatorname{Im}\phi_{1a} \subset M_1$. The subset of $\mathbf{R}^N$ consisting of a such that $p|_{(q\circ p)^{-1}(a)}$ fails to satisfy this condition is $q \circ p(X \cap \mathbf{R}^N \times (\mathbf{R}^n - M_1) \times \mathbf{R}^{n'})$ and hence is semialgebraic. Let A denote its complement in $\mathbf{R}^N$. Thus $\operatorname{Im}\phi_{1a} \subset M_1$ if and only if $a \in A$.

Next consider when $U \cap \operatorname{Im}\phi_{1a}$ is normal crossing and $\phi_{1a}|_{\phi_{1a}^{-1}(\operatorname{Reg}(U\cap\operatorname{Im}\phi_{1a}))}$ is injective. For that, remember that the tangent space T_xM_{1a}' of M_{1a}' at $x \in M_{1a}'$, for M_{1a}' described by f_i, $g_{I',i''}, ...$ as above, is given by

$$T_xM_{1a}' = \{v \in \mathbf{R}^{n+n'} : df_{1x}v = \cdots = df_{lx}v = 0\}$$

and hence the set TX defined to be $\{(a,x,y) \in X \times \mathbf{R}^{n+n'} : y \in T_x M'_{1a}\}$ is semialgebraic. Assume that $a \in A$. Set

$$M''_{1a} = \{(x,x') \in M'_{1a} \times M'_{1a} : x \neq x',\ \phi_{1a}(x) = \phi_{1a}(x') \in U,$$
$$\dim(d\phi_{1ax}(T_x M'_{1a}) + d\phi_{1ax'}(T_{x'} M'_{1a})) = \dim M - 1\}.$$

Then M''_{1a} and $\cup_{a\in A}\{a\} \times M''_{1a}$ are semialgebraic, and $a \in A - q'(\cup_{a\in A}\{a\} \times M''_{1a})$ if and only if for any $x \neq x' \in M'_{1a}$ with $\phi_{1a}(x) \in U$, the germs of ϕ_{1a} at x and x' intersect transversally, where $q' : \mathbf{R}^N \times \mathbf{R}^n \times \mathbf{R}^{n'} \times \mathbf{R}^n \times \mathbf{R}^{n'} \to \mathbf{R}^N$ denotes the projection. Repeating the same arguments on m-tuple of M'_{1a} for any $m \leq \dim M$ we obtain a semialgebraic subset B of A such that for $a \in A$, then $a \in B$ if and only if $U \cap \operatorname{Im}\phi_{1a}$ is normal crossing and $\phi_{1a}|_{\phi_{1a}^{-1}(\mathrm{Reg}(U\cap \operatorname{Im}\phi_{1a}))}$ is injective.

Let $\{B_i\}$ be a finite stratification of B into connected Nash manifolds such that $q \circ p$ is Nash trivial over each B_i [C-S$_1$], i.e., for each i there exists a Nash diffeomorphism $\pi_i : (q\circ p)^{-1}(B_i) \to (q \circ p)^{-1}(b_i) \times B_i$ of the form $\pi_i = (\pi'_i, q\circ p)$ for some $b_i \in B_i$. For $a \in B$, set $M'_a = \phi_{1a}^{-1}(M)$ and $\phi_a = \phi_{1a}|_{M'_a}$. Then $\{\phi_a : M'_a \to \mathbf{R}^n\}_{a\in B}$ is the family of all $\phi : M' \to M$ as in lemma 4.4 which are extensible to $\phi_1 : M'_1 \to M_1$ with fixed U and complexity l, d, n', and if a and a' are in the same $B_i,\ i \in I$, then $\phi_a : M'_a \to M$ and $\phi_{a'} : M'_{a'} \to M$ are Nash R-L equivalent by lemma 4.3 for the following reason. As there exists a C^0 curve in B_i joining a and a', considering a finite sequence of points on the curve we can assume that a' is close to a as elements of $\mathbf{R}^N$. We can replace ϕ_a and $\phi_{a'}$ with $\phi_a \circ (\pi'_i|_{\{a\}\times M'_a})^{-1} = p_n \circ (\pi'_i|_{\{a\}\times M'_a})^{-1} : \{b_i\} \times M'_{b_i} \to \mathbf{R}^n$ and $p_n \circ (\pi'_i|_{\{a'\}\times M'_{a'}})^{-1} : \{b_i\} \times M'_{b_i} \to \mathbf{R}^n$, where p_n denotes the projection $\mathbf{R}^N \times \mathbf{R}^n \times \mathbf{R}^{n'} \to \mathbf{R}^n$. Hence in order to apply lemma 4.3 it suffices to see that $(\pi'_i|_{\{a'\}\times M'_{a'}})^{-1}$ is close $(\pi'_i|_{\{a\}\times M'_a})^{-1}$ in the C^1 topology. That is true because we can regard $(\pi'_i|_{\{a'\}\times M'_{a'}})^{-1}$ and $(\pi'_i|_{\{a\}\times M'_a})^{-1}$ as $\pi_i^{-1}|_{(q\circ p)^{-1}(b_i)\times\{a'\}}$ and $\pi_i^{-1}|_{(q\circ p)^{-1}(b_i)\times\{a\}}$, respectively, $(q\circ p)^{-1}(b_i)$ is compact and because of the following fact. For compact C^1 manifolds M_2 and M_3 and for a C^1 function $g : M_2 \times M_3 \to \mathbf{R}$ if two points u and v in M_3 are close each other then the functions $M_2 \ni x \to g(x,u) \in \mathbf{R}$ and $M_2 \ni x \to g(x,v) \in \mathbf{R}$ are close in the C^1 topology. Hence the cardinality of equivalence classes of $\phi_a : M'_a \to M,\ a \in B$, is finite. Until now we have fixed U. We need argue for all semialgebraic neighborhood U of M in M_1. However, it is sufficient to treat a countable number of U's since M is compact. Thus the cardinality of all equivalence classes is countable. $\square$

4.2. Compactification of a Nash function with only normal crossing singularities.

The following lemmas 4.5, 4.6, 4.7 and proposition 4.8 are preparations for proposition 4.9 that states the compactification of a Nash function with only normal crossing singularities. The main tools are Nash sheaf theory and the Nash version of Hironaka desingularization theorems.

Lemmas 4.5 and 4.6 show that a normal crossing Nash subset of a non-compact Nash manifold is trivial at infinity.

Lemma 4.5. *Let X be a normal crossing Nash subset of a Nash manifold M and $f : M \to \mathbf{R}^m$ a proper Nash map whose restrictions to $M - X$ and to strata of the canonical stratification of X are submersions onto $\mathbf{R}^m$. Then f is Nash trivial, i.e., there exists a Nash diffeomorphism $\pi : M \to f^{-1}(0) \times \mathbf{R}^m$ of the form $\pi = (\pi', f)$, and π' can be chosen so that $\pi'(X) = X \cap f^{-1}(0)$ and $\pi' = \mathrm{id}$ on $f^{-1}(0)$.*

The analytic case also holds.

This is shown in [C-S$_{1,2}$] in the case of empty X. We prove here the nonempty case.

Proof of lemma 4.5. Consider the Nash case. Let $n = \dim M$, take k an integer smaller than n, and let X_k denote the union of strata of the canonical stratification of X of dimension less than or equal to k. We define π' on X_k by induction on k, and then on M. To this aim, we can assume that π' is already given on X, say $\pi_X = (\pi'_X, f_X)$, by the following fact, where $f_A = f|_A$ for a subset A of M.

Fact 1. There exist a Nash manifold $\tilde{X}_k$ of dimension k and a Nash immersion $p_{\tilde{X}_k} : \tilde{X}_k \to M$ such that $\operatorname{Im} p_{\tilde{X}_k} = X_k$ and $p_{\tilde{X}_k}|_{p^{-1}_{\tilde{X}_k}(X_k - X_{k-1})}$ is injective.

Proof of fact 1. (Artin-Mazur Theorem. See the proof of theorem 2.9.) Let M be contained and closed in $\mathbf{R}^N$, and let X_k^Z denote the Zariski closure of X_k in $\mathbf{R}^N$. Then there exist an algebraic variety $\widetilde{X_k^Z}$ (the normalization of X_k^Z) in $\mathbf{R}^N \times \mathbf{R}^{N'}$ for some $N' \in \mathbf{N}$ and the union of some connected components $\tilde{X}_k$ of $\widetilde{X_k^Z}$ such that $\widetilde{X_k^Z}$ is non-singular at $\tilde{X}_k$. Hence $\tilde{X}_k$ is a Nash manifold and the restriction $p_{\tilde{X}_k}$ to $\tilde{X}_k$ of the projection $p : \mathbf{R}^N \times \mathbf{R}^{N'} \to \mathbf{R}^N$ satisfies the requirements in fact 1.

Let ϕ_i be a finite number of global generators of the sheaf of $\mathcal{N}$-ideals $\mathcal{I}$ on M defined by X, and set $\phi = \sum \phi_i^2$. Then $\phi > 0$ on $M - X$ and ϕ is a global generator of $\mathcal{I}^2$. For a subset A of M and $x \in \mathbf{R}^m$, set $A(x) = A \cap f^{-1}(x)$. We will extend π'_X to π'. For that it suffices to find π' of class semialgebraic C^l for a large integer l for the following reason.

Fact 2. Let g be a semialgebraic C^l function on M whose restriction to X is of class Nash. Then fixing g on X we can approximate g by a Nash function in the semialgebraic C^{l-n} topology.

Proof of fact 2. As in the proof of theorem 3.1,(1), step 1, $g|_X$ is extensible to a Nash function G on M by theorem 2.8. Replace g with $g - G$. Then we can assume that $g = 0$ on X and g is of the form $\sum g_i\phi_i$ for some semialgebraic C^{l-n} functions g_i on M for the following reason. Reduce the problem to the case where $(M, X) = (\mathbf{R}^n, \{x_1 \cdots x_{n'} = 0\})$ and $\{\phi_i\} = \{x_1 \cdots x_{n'}\}$ for some $n' \in \mathbf{N}$ by a partition of unity of class semialgebraic C^l (remark 2.11,(5)$'$). Then g is divisible by $x_1 \cdots x_{n'}$ as a semialgebraic C^{l-n} function on M by elementary calculations. Hence g is of the form $g_1 x_1 \cdots x_{n'}$ for some semialgebraic C^{l-n} function g_1 on M. As usual, we approximate g_i by Nash functions $\tilde{g}_i$ in the semialgebraic C^{l-n} topology we obtain the required approximation $\sum \tilde{g}_i\phi_i$ of g in fact 2.

We will see that there exists a finite semialgebraic C^l stratification $\{B_i\}$ of $\mathbf{R}^m$ such that for each i, the map $\pi_X|_{X \cap f^{-1}(B_i)}$ is extensible to a semialgebraic C^l diffeomorphism $\pi_i = (\pi'_i, f_{f^{-1}(B_i)}) : f^{-1}(B_i) \to M(b_i) \times B_i$ for some point $b_i \in B_i$. In the following arguments we need to stratify $\mathbf{R}^m$ into $\{B_i\}$, each B_i into $\{B_{i,j} : j = 1, 2, ...\}$ and once more. However, we always use notation $\mathbf{R}^m$ for all strata for simplicity of notation, which does not cause problems because we can choose stratifications so that strata are semialgebraically C^l diffeomorphic to Euclidean spaces.

We recall the construction of π as in the proof of Theorem II.6.7 in [S$_3$]. Without loss of generality we assume that $\pi'_X|_{X(0)} = \mathrm{id}$. First we can modify in order ϕ to be a semialgebraic C^l function so that for each $x \in \mathbf{R}^m$, $\phi|_{M(x)-X}$ has only singularities of Morse type (Claim 2, ibid.). (here we need to stratify $\mathbf{R}^m$ and

consider the restriction of ϕ to each stratum in place of ϕ, and the main method of proof is a semialgebraic version of Thom's transversality theorem), ϕ is constant on each connected component of $Z \overset{\text{def}}{=} \cup_{x\in\mathbf{R}^m} \operatorname{Sing}(\phi|_{M(x)-X})$ and the values are distinct from each other (Claim 4, ibid.) after the second stratification. Next, let Y be a connected component of Z and set $\tilde{Y} = \phi^{-1}(\phi(Y))$. Then there exist an open semialgebraic neighborhood U of $\tilde{Y}$ in M and a semialgebraic C^l embedding $u = (u', f_U) : U \to U(0) \times \mathbf{R}^m$ such that $u'|_{U(0)} = \mathrm{id}$ and $\phi \circ u' = \phi|_U$ (Claim 5, ibid.) after the third stratification. Thirdly, applying lemma 4.5 without X to the semialgebraic C^l map $(f, \phi)|_{\phi^{-1}(I)} : \phi^{-1}(I) \to \mathbf{R}^m \times I$ for each connected component I of $(0, \infty) - \phi(Z)$, we obtain a semialgebraic C^l diffeomorphism $\lambda = (\lambda', f_{\phi^{-1}(I)}, \phi|_{\phi^{-1}(I)}) : \phi^{-1}(I) \to \phi^{-1}(I)(0,0) \times \mathbf{R}^m \times I$ such that $\lambda'|_{\phi^{-1}(I)(0,0)} = \mathrm{id}$, where $\phi^{-1}(I)(0,0) = \phi^{-1}(t_0) \cap M(0)$ for some $t_0 \in I$ (Claim 7, ibid.). Fourthly, we paste u and λ for all I and construct a semialgebraic C^l diffeomorphism $v = (v', f_{M-X}) : M - X \to (M(0) - X) \times \mathbf{R}^m$ such that $v'|_{M(0)-X} = \mathrm{id}$ and $\phi \circ v' = \phi|_{M-X}$, ibid. Hence it suffices to prove the following fact by the same idea of pasting.

Fact 3. There exist an open semialgebraic neighborhood W of X in M and a semialgebraic C^l embedding $w = (w', f_W) : W \to M(0) \times \mathbf{R}^m$ such that $w' = \pi'_X$ on X, so that $w'|_{W(0)} = \mathrm{id}$ and $\phi \circ w' = \phi|_W$.

Proof of fact 3. Here the condition $\phi \circ w' = \phi|_W$ is not necessary. If there exists a semialgebraic C^l embedding w without this condition, we replace ϕ on W with $\phi \circ w'$, extend it to a semialgebraic C^l function on M positive on $M - X$, and repeat the above arguments from the beginning. Then fact 3 is satisfied by this w'.

If X is smooth, the problem becomes easier. Hence we reduce to the smooth case. Let $\tilde{X} \subset \mathbf{R}^N \times \mathbf{R}^{N'}$ and $p_{\tilde{X}} : \tilde{X} \to M$ be a Nash manifold and the restriction to $\tilde{X}$ of the projection $p : \mathbf{R}^N \times \mathbf{R}^{N'} \to \mathbf{R}^N$ defined in the proof of fact 1 for $k = n - 1$. For a small positive semialgebraic C^0 function ϵ on $\tilde{X}$, let $\tilde{Q}$ denote the ϵ-neighborhood of $\tilde{X}$ in $M \times \mathbf{R}^{N'}$, i.e.,

$$\tilde{Q} = \bigcup_{z\in\tilde{X}} \{z' \in M \times \mathbf{R}^{N'} : \operatorname{dis}(z, z') < \epsilon(z)\},$$

and let $\tilde{q} : \tilde{Q} \to \tilde{X}$ denote the orthogonal projection, which is a Nash submersion. Set

$$\tilde{M} = \{(x, y) \in \tilde{Q} \subset M \times \mathbf{R}^{N'} : \tilde{q}(x, y) = (x', y) \text{ for some } x' \in X\}.$$

Then $\tilde{M}$ is a Nash manifold of dimension n containing $\tilde{X}$, and $p_{\tilde{M}} : \tilde{M} \to M$ is a (not necessarily proper) Nash immersion, where p_A denotes $p|_A$ for a subset A of $M \times \mathbf{R}^{N'}$. Set $A(0) = A \cap M(0) \times \mathbf{R}^{N'}$, set $f_A = f \circ p_A$ for the same A, and $\tilde{\tilde{X}} = p_{\tilde{M}}^{-1}(X)$. Then $\tilde{\tilde{X}}$ is a normal crossing Nash subset of $\tilde{M}$, and $p_{\tilde{\tilde{X}}} : \tilde{\tilde{X}} \to X$ is a (not necessarily proper) local Nash diffeomorphism at each point of $\tilde{\tilde{X}}$. Moreover $\pi_X = (\pi'_X, f_X)$ is lifted to $\pi_{\tilde{\tilde{X}}} = (\pi'_{\tilde{\tilde{X}}}, f_{\tilde{\tilde{X}}}) : \tilde{\tilde{X}} \to \tilde{\tilde{X}}(0) \times \mathbf{R}^m$, and there exists a Nash function $\tilde{\phi}$ on $\tilde{M}$ with zero set $\tilde{X}$ which is, locally at each point of $\tilde{X}$, the square of a regular function and such that

$$\tilde{\phi} = \tilde{\phi} \circ \pi'_{\tilde{\tilde{X}}} \quad \text{on } \tilde{\tilde{X}} \tag{1}$$

To be precise, we construct $\tilde\phi$ first on $\tilde M(0)$, and extend it to $\tilde{\tilde X}$ so that (1) is satisfied and then to $\tilde M$ as usual. Moreover $\pi'_{\tilde{\tilde X}} = \mathrm{id}$ on $\tilde{\tilde X}(0)$.

Note that $X_{m-1} = \emptyset$ since $f_{X_k - X_{k-1}}$ is a submersion onto $\mathbf{R}^m$ if $X_k \neq \emptyset$. Let $m \le k < n$. Then by the definition of $\tilde X$, the map $p_{\tilde X \cap p^{-1}(X_k - X_{k-1})} : \tilde X \cap p^{-1}(X_k - X_{k-1}) \to X_k - X_{k-1}$ is a Nash $(n-k)$-fold covering. Hence considering a semialgebraic triangulation of $X_k(0)$ compatible with X_{k-1}—a semialgebraic homeomorphism from the underlying polyhedron of some simplicial complex to $X_k(0)$ such that $X_{k-1}(0)$ is the image of the union of some simplexes—and small open semialgebraic neighborhoods of inverse image of open simplexes by $\pi_X'^{-1}$ in $M - X_{k-1}$, we obtain finite open semialgebraic coverings $\{Q_{k,i} : i\}$ of $X_k - X_{k-1}$ in $M - X_{k-1}$ and $\{\tilde Q_{k,i,j} : i,\ 1 \le j \le n-k\}$ of $\tilde X \cap p^{-1}(X_k - X_{k-1})$ in $\tilde M - \tilde X \cap p^{-1}(X_{k-1})$ such that $\pi_X'^{-1}(X(0) \cap Q_{k,i}) = X \cap Q_{k,i}$, $(Q_{k,i}, X_k \cap Q_{k,i})$ are Nash diffeomorphic to $(\mathbf{R}^n, \{0\} \times \mathbf{R}^k)$, such that $p_{\tilde Q_{k,i,j}} : (\tilde Q_{k,i,j}, \tilde X \cap p^{-1}(X_k) \cap \tilde Q_{k,i,j}) \to (Q_{k,i}, X_k \cap Q_{k,i})$ are Nash diffeomorphisms, and $\tilde Q_{k,i,j} \cap \tilde Q_{k,i,j'} = \emptyset$ if $j \neq j'$. Define Nash functions $\phi_{k,i,j}$ on $Q_{k,i}$ by $\tilde\phi \circ p^{-1}_{\tilde Q_{k,i,j}}$. Then $\phi_{k,i,j}$ are the squares of Nash functions, say $\phi^{1/2}_{k,i,j}$, and we can choose $Q_{k,i}$ so small that the maps $(f, \phi^{1/2}_{k,i,1}, ..., \phi^{1/2}_{k,i,n-k}) : Q_{k,i} \to \mathbf{R}^{m+n-k}$ are submersions, that if $Q_{k,i} \cap Q_{k,i'} \neq \emptyset$ then

$$(2) \qquad \{\phi_{k,i,j}|_{Q_{k,i} \cap Q_{k,i'}} : j = 1, ..., n-k\} = \{\phi_{k,i',j}|_{Q_{k,i} \cap Q_{k,i'}} : j = 1, ..., n-k\},$$

and that if $Q_{k,i} \cap Q_{k',i'} \neq \emptyset$ for $k < k'$ then

$$(3) \quad \{\phi_{k,i,j}|_{Q_{k,i} \cap Q_{k',i'}} : j = 1, ..., n-k\} \supset \{\phi_{k',i',j}|_{Q_{k,i} \cap Q_{k',i'}} : j = 1, ..., n-k'\}.$$

Let $\Phi_{k,k',i,i'}$ denote the $k'-k$ Nash functions on $Q_{k,i} \cap Q_{k',i'}$ in the complement in (3). Note that (1) implies

$$(1)' \qquad \phi^{1/2}_{k,i,j} \circ \pi'_X = \phi^{1/2}_{k,i,j} \quad \text{on } X \cap Q_{k,i}.$$

We work from now in the semialgebraic C^l category. Shrink again $Q_{k,i}$ (fixing always $X_k \cap Q_{k,i}$), and set $Q_k = \cup_i Q_{k,i}$. Then there exist semialgebraic C^l submersive retractions $q_k : Q_k \to X_k - X_{k-1}$ such that

$$(4) \qquad f \circ q_k = f \quad \text{on } Q_k,$$

$$(5) \qquad q_k \circ \pi'_X = \pi'_X \circ q_k \quad \text{on } X \cap Q_k,$$

and the maps $(q_k|_{Q_{k,i}}, \phi^{1/2}_{k,i,1}, ..., \phi^{1/2}_{k,i,n-k}) : Q_{k,i} \to (X_k - X_{k-1}) \times \mathbf{R}^{n-k}$ are semialgebraic C^l embeddings as follows.
For a while, assume that q_k on $Q_k(0)$ are already given so that the following controlled conditions are satisfied.

$$(6)_0 \qquad q_k \circ q_{k'} = q_k \quad \text{on } Q_k(0) \cap Q_{k'}(0) \text{ for } k < k',$$

$(7)_0$

$\phi^{1/2} \circ q = \phi^{1/2}$ on Q $(0) \cap Q$ (0) for $k < k'$ and ϕ $|$ $\subset \Phi$

Extend each q_k to $q_k : X \cap Q_k \to X_k - X_{k-1}$ so that (4) and (5) are satisfied as follows, which is uniquely determined, though we need to choose Q_k so that $\pi'_X(X \cap Q_k) \subset Q_k(0)$. For $(x, y) \in Q_k^2$ with small $\mathrm{dis}(x, y)$, let $r(x, y)$ denote the orthogonal projection image of x to $X_k(f(y)) - X_{k-1}$. Let $q'_k : Q_k \to X_k - X_{k-1}$ be any semialgebraic C^l extension of q_k, shrink Q_k and define $q_k(x)$ for $x \in Q_k$ to be $r(q'_k(x), x)$. Then q_k is a semialgebraic C^l submersive retraction of Q_k to $X_k - X_{k-1}$ and satisfies (4) and (5).
Hence $(\phi^{1/2}_{k,i,1}, ..., \phi^{1/2}_{k,i,n-k})$ is a local coordinate system of $q_k^{-1}(x) \cap Q_{k,i}$ at x, for each $x \in X_k - X_{k-1}$. Therefore, by (3), for each $Q_{k,i}$ and $Q_{k',i'}$ with $k < k'$ there exists a unique semialgebraic C^l submersion $q_{k,k',i,i'} : Q_{k,i} \cap Q_{k',i'} \to X_{k'} \cap Q_{k,i}$ such that

$$q_k \circ q_{k,k',i,i'} = q_k \quad \text{on } Q_{k,i} \cap Q_{k',i'} \text{ and} \tag{8}$$

$$\phi^{1/2}_{k,i,j} \circ q_{k,k',i,i'} = \phi^{1/2}_{k,i,j} \quad \text{on } Q_{k,i} \cap Q_{k',i'} \text{ for } \phi_{k,i,j}|_{Q_{k,i} \cap Q_{k',i'}} \in \Phi_{k,k',i,i'}. \tag{9}$$

To be precise, the domain of definition of $q_{k,k',i,i'}$ is $q_{k'}^{-1}(Q_{k,i}) \cap Q_{k,i} \cap Q_{k',i'}$. However, we omit $q_{k'}^{-1}(Q_{k,i})$ for simplicity of notation. In the following arguments also we simplify the domains of definition of many maps. By the above equalities we have the following equalities $(4)'$, $(5)'$ and (10).

$$f \circ q_{k,k',i,i'} \overset{(4)}{=} f \circ q_k \circ q_{k,k',i,i'} \overset{(8)}{=} f \circ q_k \overset{(4)}{=} f \quad \text{on } Q_{k,i} \cap Q_{k',i'}. \tag*{$(4)'$}$$

$$q_k \circ q_{k,k',i,i'} \circ \pi'_X \overset{(8)}{=} q_k \circ \pi'_X \overset{(5)}{=} \pi'_X \circ q_k \overset{(8)}{=} \pi'_X \circ q_k \circ q_{k,k'i,i'} \overset{(5)}{=} q_k \circ \pi'_X \circ q_{k,k',i,i'}$$
$$\text{on } X \cap Q_{k,i} \cap Q_{k',i'},$$

$$\phi^{1/2}_{k,i,j} \circ q_{k,k',i,i'} \circ \pi'_X \overset{(9)}{=} \phi^{1/2}_{k,i,j} \circ \pi'_X \overset{(1)'}{=} \phi^{1/2}_{k,i,j} \overset{(9)}{=} \phi^{1/2}_{k,i,j} \circ q_{k,k'i,i'} \overset{(1)'}{=} \phi^{1/2}_{k,i,j} \circ \pi'_X \circ q_{k,k',i,i'}$$
$$\text{on } X \cap Q_{k,i} \cap Q_{k',i'} \text{ for } \phi_{k,i,j}|_{Q_{k,i} \cap Q_{k',i'}} \in \Phi_{k,k'i,i'},$$

hence by embeddingness of $(q_k|_{Q_{k,i}}, \phi^{1/2}_{k,i,1}, ..., \phi^{1/2}_{k,i,n-k})$

$$q_{k,k',i,i'} \circ \pi'_X = \pi'_X \circ q_{k,k',i,i'} \quad \text{on } X \cap Q_{k,i} \cap Q_{k',i'}. \tag*{$(5)'$}$$

By assumption, $(6)_0$ and $(7)_0$ hold. Then by (4) and (5)

$$q_k \circ q_{k'} = q_k \quad \text{on } X \cap Q_k \cap Q_{k'} \text{ for } k < k', \tag*{$(6)_X$}$$

$(7)_X$
$$\phi^{1/2}_{k,i,j} \circ q_{k'} = \phi^{1/2}_{k,i,j} \text{ on } X \cap Q_{k,i} \cap Q_{k',i'} \text{ for } k < k' \text{ and } \phi_{k,i,j}|_{Q_{k,i} \cap Q_{k',i'}} \in \Phi_{k,k',i,i'}.$$

Hence by the same embeddingness as above

$$q_{k,k',i,i'} = q_{k'} \quad \text{on } X \cap Q_{k,i} \cap Q_{k',i'}. \tag{10}$$

Compare q_{k,k',i_1,i'_1} and q_{k,k',i_2,i'_2}. By (2) and (3)

$$q_{k,k',i_1,i'_1} = q_{k,k',i_1,i'_2} \quad \text{on } Q_{k,i_1} \cap Q_{k',i'_1} \cap Q_{k',i'_2},$$
$$q_{k,k',i_1,i'_2} = q_{k,k',i_2,i'_2} \quad \text{on } Q_{k,i_1} \cap Q_{k,i_2} \cap Q_{k',i'_2},$$

hence

Therefore, we have semialgebraic C^l submersions $q_{k,k'}: Q_k \cap Q_{k'} \to X_{k'} - X_{k'-1}$, $k < k'$, such that

$$f \circ q_{k,k'} = f \quad \text{on } Q_k \cap Q_{k'}, \tag{4$'$}$$

$$q_k \circ q_{k,k'} = q_k \quad \text{on } Q_k \cap Q_{k'}, \tag{8}$$

$$\phi_{k,i,j}^{1/2} \circ q_{k,k'} = \phi_{k,i,j}^{1/2} \quad \text{on } Q_{k,i} \cap Q_{k',i'} \text{ for } \phi_{k,i,j}|_{Q_{k,i}\cap Q_{k',i'}} \in \Phi_{k,k',i,i'}, \tag{9}$$

$$q_{k,k'} = q_{k'} \quad \text{on } X \cap Q_k \cap Q_{k'}. \tag{10}$$

We want to shrink the Q_k's and modify the q_k's keeping (4) and (5) so that

$$q_{k,k'} = q_{k'} \quad \text{on } Q_k \cap Q_{k'} \text{ for } k < k'. \tag{11}$$

We proceed by double induction. Let $m \le k_1 < k_2 < n \in \mathbf{N}$, and assume that (11) holds for $k < k' < k_2$ and for $k_1 < k < k' = k_2$. Fix q_k and $k < k_2$. Then we need to modify q_{k_2} so that (11) holds for $k = k_1$ and $k' = k_2$. Let ξ be a semialgebraic C^l function on $M - X_{k_1-1}$ such that $0 \le \xi \le 1$, and $\xi = 1$ outside of a small open semialgebraic neighborhood Q'_{k_1} $(\subset Q_{k_1})$ of $X_{k_1} - X_{k_1-1}$ in $M - X_{k_1-1}$ and moreover $\xi = 0$ on a smaller one Q''_{k_1}. Shrink Q_{k_2} and define a semialgebraic C^l submersive retraction $q'_{k_2}: Q_{k_2} \to X_{k_2} - X_{k_2-1}$ by $q'_{k_2} = q_{k_2}$ on $Q_{k_2} - Q'_{k_1}$ and for $x \in Q_{k_2} \cap Q'_{k_1}$, let $q'_{k_2}(x)$ be the orthogonal projection image of $\xi(x)q_{k_2}(x) + (1-\xi(x))q_{k_1,k_2}(x) \in \mathbf{R}^N$ to the Nash manifold $X_{k_2}(f(x)) - X_{k_2-1}$. Then q'_{k_2} satisfies (4) and (11) for $k = k_1$, $k' = k_2$ and for Q_k replaced by Q''_{k_1}, the map $(q'_{k_2}|_{Q_{k_2,i}}, \phi_{k_2,i,1}^{1/2}, ..., \phi_{k_2,i,n-k_2}^{1/2}): Q_{k_2,i} \to (X_{k_2 - X_{k_2-1}}) \times \mathbf{R}^{n-k_2}$ continues to be a semialgebraic C^l embedding if we shrink $Q_{k_2,i}$ (of course, fixing $Q_{k_2,i} \cap X_{k_2}$), $q'_{k_2} = q_{k_2}$ on $X \cap Q_{k_2}$ by (10), hence (5) for q'_{k_2} holds, and $q'_{k_2} = q_{k_2}$ on $Q_{k_2} \cap \cup_{k_1<k<k_2} Q_k$ for the following reason. Let $k_1 < k_3 < k_2$. It suffices to see that $q_{k_1,k_2} = q_{k_2}$ on $Q_{k_1} \cap Q_{k_2} \cap Q_{k_3}$, which is equivalent, by uniqueness of q_{k_1,k_2}, to

$$q_{k_1} \circ q_{k_2} = q_{k_1} \quad \text{on } Q_{k_1} \cap Q_{k_2} \cap Q_{k_3}, \tag{12}$$

$$\begin{aligned} \phi_{k_1,i_1,j}^{1/2} \circ q_{k_2} = \phi_{k_1,i_1,j}^{1/2} \quad &\text{on } Q_{k_1,i_1} \cap Q_{k_2,i_2} \cap Q_{k_3} \\ &\text{for } \phi_{k_1,i_1,j}|_{Q_{k_1,i_1}\cap Q_{k_2,i_2}} \in \Phi_{k_1,k_2,i_1,i_2}. \end{aligned} \tag{13}$$

By (8) for $k = k_1$ and $k' = k_3$ and for $k = k_3$ and $k' = k_2$

$$\begin{aligned} q_{k_1} \circ q_{k_1,k_3} &= q_{k_1} \quad \text{on } Q_{k_1} \cap Q_{k_3}, \\ q_{k_3} \circ q_{k_3,k_2} &= q_{k_3} \quad \text{on } Q_{k_2} \cap Q_{k_3}. \end{aligned}$$

By (11) for $k = k_1$ and $k' = k_3$ and for $k = k_3$ and $k' = k_2$

$$\begin{aligned} q_{k_1,k_3} &= q_{k_3} \quad \text{on } Q_{k_1} \cap Q_{k_3}, \\ q_{k_3,k_2} &= q_{k_2} \quad \text{on } Q_{k_2} \cap Q_{k_3}. \end{aligned}$$

Hence

$$q_{k_1} \circ q_{k_3} = q_{k_1} \quad \text{on } Q_{k_1} \cap Q_{k_3}, \tag{14}$$

(15)

Therefore,

$$q_{k_1} \circ q_{k_2} \overset{(14)}{=} q_{k_1} \circ q_{k_3} \circ q_{k_2} \overset{(15)}{=} q_{k_1} \circ q_{k_3} \overset{(14)}{=} q_{k_1} \quad \text{on } Q_{k_1} \cap Q_{k_2} \cap Q_{k_3}. \tag{12}$$

We can prove (13) in the same way because if $Q_{k_1,i_1} \cap Q_{k_2,i_2} \cap Q_{k_3,i_3} \neq \emptyset$ then $\Phi_{k_1,k_2,i_1,i_2}|_{Q_{k_1,i_1} \cap Q_{k_2,i_2} \cap Q_{k_3,i_3}}$ is the disjoint union of $\Phi_{k_1,k_3,i_1,i_3}|_{Q_{k_1,i_1} \cap Q_{k_2,i_2} \cap Q_{k_3,i_3}}$ and $\Phi_{k_3,k_2,i_3,i_2}|_{Q_{k_1,i_1} \cap Q_{k_2,i_2} \cap Q_{k_3,i_3}}$. Thus the induction process works, and we assume that (11) is satisfied. Consequently, the following **controlledness** conditions are satisfied by (8), (9) and (11).

$$q_k \circ q_{k'} = q_k \quad \text{on } Q_k \cap Q_{k'} \text{ for } k < k', \tag{6}$$

(7)
$$\phi^{1/2}_{k,i,j} \circ q_{k'} = \phi^{1/2}_{k,i,j} \text{ on } Q_{k,i} \cap Q_{k',i'} \text{ for } k < k' \text{ and } \phi_{k,i,j}|_{Q_{k,i} \cap Q_{k',i'}} \in \Phi_{k,k',i,i'}.$$

It remains to construct q_k on $Q_k(0)$. First define r as above, i.e., for $(x, y) \in Q_k^2(0)$ with small $\mathrm{dis}(x, y)$, let $r(x, y)$ denote the orthogonal projection image of x to $X_k(0) - X_{k-1}$. Set $q_k(x) = r(x, x)$ for $x \in Q_k(0)$. Then $q_k : Q_k(0) \to X_k(0) - X_{k-1}$ are Nash submersive retractions. We need to modify them so that $(6)_0$ and $(7)_0$ are satisfied. This is clearly possible by the above arguments.

Now we define W and w as in fact 3. Set $W = \cup_{k=m}^{n-1} Q_k$ and consider each $Q_{k,i}$. Shrink $Q_{k,i}$ so that

$$(\pi'_X \circ q_k, \phi^{1/2}_{k,i,1}, ..., \phi^{1/2}_{k,i,n-k})(Q_{k,i}) \subset (q_k, \phi^{1/2}_{k,i,1}, ..., \phi^{1/2}_{k,i,n-k})(Q_{k,i}(0)).$$

Then for each $x \in Q_{k,i}$ there exists a unique $y \in Q_{k,i}(0)$ such that

$$(q_k, \phi^{1/2}_{k,i,1}, ..., \phi^{1/2}_{k,i,n-k})(y) = (\pi'_X \circ q_k, \phi^{1/2}_{k,i,1}, ..., \phi^{1/2}_{k,i,n-k})(x).$$

The correspondence $w'_{k,i}$ from x to y is a semialgebraic C^l map such that $w_{k,i} = (w'_{k,i}, f_{Q_{k,i}}) : Q_{k,i} \to Q_{k,i}(0) \times \mathbf{R}^m$ is a semialgebraic C^l embedding by (4), $w'_{k,i} = \pi'_X$ on $X \cap Q_{k,i}$ because

$$(q_k, \phi^{1/2}_{k,i,1}, ..., \phi^{1/2}_{k,i,n-k}) \circ \pi'_X(x) \overset{(1)',(5)}{=} (\pi'_X \circ q_k, \phi^{1/2}_{k,i,1}, ..., \phi^{1/2}_{k,i,n-k})(x) \text{ for } x \in X \cap Q_{k,i},$$

and $w'_{k,i}|_{Q_{k,i}(0)} = \mathrm{id}$ by (4) and by the equality $\pi'_X = \mathrm{id}$ on $X(0)$. Hence it suffices to see that $w'_{k,i} = w'_{k',i'}$ on $Q_{k,i} \cap Q_{k',i,}$. This is clear by (2) if $k = k'$ and $Q_{k,i} \cap Q_{k',i'} \neq \emptyset$. Assume that $k < k'$ and $Q_{k,i} \cap Q_{k',i'} \neq \emptyset$. By (3) we suppose that

$$\phi^{1/2}_{k',i',j} = \phi^{1/2}_{k,i,j+k'-k} \quad \text{on } Q_{k,i} \cap Q_{k',i'},\ j = 1, ..., n - k'.$$

Then by the definition of $w'_{k,i}$ and $w'_{k',i'}$ we only need to show that

$$q_{k'} \circ w'_{k,i} = \pi'_X \circ q_{k'} \quad \text{on } Q_{k,i} \cap Q_{k',i'},$$

which is equivalent to

$$q_k \circ q_{k'} \circ w'_{k,i} = q_k \circ \pi'_X \circ q_{k'} \quad \text{on } Q_{k,i} \cap Q_{k',i'} \text{ and}$$
$$\phi^{1/2}_{k,i,j} \circ q_{k'} \circ w'_{k,i} = \phi^{1/2}_{k,i,j} \circ \pi'_X \circ q_{k'} \quad \text{on } Q_{k,i} \cap Q_{k',i'},\ j = 1, ..., k' - k$$

We have

$$q_k \circ q_{k'} \circ w'_{k,i} \overset{(6)}{=} q_k \circ w'_{k,i} = \pi'_X \circ q_k \overset{(6)}{=} \pi'_X \circ q_k \circ q_{k'} \overset{(5)}{=} q_k \circ \pi'_X \circ q_{k'},$$
$$\phi^{1/2}_{k,i,j} \circ q_{k'} \circ w'_{k,i} \overset{(7)}{=} \phi^{1/2}_{k,i,j} \circ w'_{k,i} = \phi^{1/2}_{k,i,j} \overset{(7)}{=} \phi^{1/2}_{k,i,j} \circ q_{k'} \overset{(1)'}{=} \phi^{1/2}_{k,i,j} \circ \pi'_X \circ q_{k'},$$
$$j = 1, ..., k' - k.$$

Thus we have completed the proof of fact 3 and hence of the construction of $\pi_i = (\pi', f_{f^{-1}(B_i)})\colon f^{-1}(B_i) \to M(b_i) \times B_i$.

Next we will extend π_i to a neighborhood of $f^{-1}(B_i)$ in M. Let $\eta_i : U_i \to B_i$ be a semialgebraic submersive C^l retraction of a small semialgebraic open neighborhood of B_i in R^n. Then we only need to lift η_i to a semialgebraic submersive C^l retraction $\tilde{\eta}_i : f^{-1}(U_i) \to f^{-1}(B_i)$ so that $\tilde{\eta}_i^{-1}(X_k) = X_k \cap f^{-1}(U_i)$ for each k and $\pi'_X \circ \tilde{\eta}_i = \pi'_X$ on $X \cap f^{-1}(U_i)$ because if $\tilde{\eta}_i$ exists, the map $f^{-1}(U_i) \ni x \to (\pi'_i \circ \tilde{\eta}_i(x), f(x)) \in M(b_i) \times U_i$ is the required extension of π_i. We proceed by two steps. First we define $\tilde{\eta}_i$ on $X \cap f^{-1}(U_i)$ and then extend it to $f^{-1}(U_i)$.

The first step. We can assume $b_i = 0$. Then $\pi'_i = \pi'_X$ on $X \cap f^{-1}(B_i)$. Hence there exists a unique semialgebraic C^l diffeomorphism $\tilde{\eta}_{i,y}$ from $X \cap f^{-1}(y)$ to $X \cap f^{-1}(\eta_i(y))$ for each $y \in U_i$ such that $\pi'_i \circ \tilde{\eta}_{i,y} = \pi'_X$ on $X \cap f^{-1}(y)$. Define $\tilde{\eta} : X \cap f^{-1}(U_i) \to X \cap f^{-1}(B_i)$ by $\tilde{\eta}_i(x) = \tilde{\eta}_{i,f(x)}(x)$. Then $\tilde{\eta}_i$ satisfies the requirements.

The second step. Since B_i is Nash diffeomorphic to a Euclidean space we can regard U_i as $B_i \times \mathbf{R}^{m'}$ and η_i as the projection $\eta_i : B_i \times \mathbf{R}^{m'} \to B_i$, where $m' = m - \dim B_i$. Then we define $\tilde{\eta}_i$ on $f^{-1}(B_i \times \mathbf{R}^k \times \{0\})$ by induction on $k = 0, ..., m'$. For that it suffices to consider the case $m' = 1$. Moreover we replace $\mathbf{R}$ of $B_i \times \mathbf{R}$ with the circle $S^1 = \{x \in \mathbf{R}^2 : |x| = 1\}$ as follows. Let $\omega_i : S^1 \to \mathbf{R}$ be a Nash function such that 0 is a regular value. Let $\check{\eta}_i : B_i \times \mathbf{R} \to \mathbf{R}$ be the projection, $\hat{M}$ be the fiber product of $\check{\eta}_i \circ f : f^{-1}(U_i) \to \mathbf{R}$ and $\omega_i : S^1 \to \mathbf{R}$, $\hat{X}$ be the inverse image of $X \cup f^{-1}(B_i \times \{0\})$ under the induced map $\hat{\omega}_i : \hat{M} \to M$ and $\hat{f} : \hat{M} \to B_i$ be the naturally defined projection. Then $\hat{M}$ is a Nash manifold, $\hat{f}$ is a proper Nash map, $\hat{X}$ is a normal crossing Nash subset of $\hat{M}$, and the conditions in the lemma are satisfied for $\hat{X}$, $\hat{M}$ and $\hat{f}$. Define a map $\hat{\pi}_{i,\hat{X}} = (\hat{\pi}'_{i,\hat{X}}, \hat{f}) : \hat{X} \to (\hat{X} \cap \hat{f}^{-1}(0)) \times B_i$ so that $\pi'_X \circ \hat{\omega}_i \circ \hat{\pi}'_{i,\hat{X}} = \pi'_X \circ \hat{\omega}_i$. Then $\hat{\pi}_{i,\hat{X}}$ is a uniquely determined Nash diffeomorphism, $\hat{\pi}'_{i,\hat{X}} = \mathrm{id}$ on $\hat{X} \cap \hat{f}^{-1}(0)$, and by fact 3 the map $\hat{\pi}_{i,\hat{X}}$ is extended to a semialgebraic C^l embedding $\hat{\pi}_i = (\hat{\pi}'_i, \hat{f}) : \hat{W}_i \to \hat{f}^{-1}(0) \times B_i$ for some open neighborhood $\hat{W}_i$ of $\hat{X}$ in $\hat{M}$. We can shrink U_i and $\hat{W}_i$ so that $\hat{f}^{-1}(U_i) = \hat{W}_i$ since $\hat{f}$ is proper. Hence it remains to consider the problem of lifting η_i only on $\eta_i|_{\eta^{-1}(0)} : \eta^{-1}(0) \to \{0\}$. Namely the problem is reduced to the case where $B_i = \{0\}$. This case also follows from fact 3. Thus π_i is extended to $f^{-1}(U_i)$. We keep the notation π_i for the extension.

For the construction of π we need to modify and paste π_i together. This is what [C-S$_2$] proved. To be precise, [C-S$_1$] proved local Nash triviality and [C-S$_2$] proved that the local Nash triviality implies the global Nash triviality. They treat the case without X. However, the proof in [C-S$_2$] works in the case with X (see also the proof of Theorem II.6.3, [S$_3$]). Thus we obtain π and complete the proof of lemma 4.5 in the Nash case.

The analytic case follows in the same way. $\square$

Note that the above proof shows that the lemma still holds if M is a Nash manifold with corners and if the restrictions of f to strata of the canonical stratification $\{M_k\}$ of ∂M compatible with X are also submersions onto $\mathbf{R}^m$. Here the canonical stratification $\{M_k\}$ compatible with X is defined as follows. For a semialgebraic set S, let $\operatorname{Reg} S$ denote the subset of X consisting of points x such that S_x is a Nash manifold germ of dimension $\dim S$. Then $M_{n-1} = \operatorname{Reg}(\partial M - X)$, $M_{n-2} = \operatorname{Reg}(\partial M - M_{n-1})$, $M_{n-3} = \operatorname{Reg}(\partial M - M_{n-1} - M_{n-2}), ...$ Note that $\{M_k\}$ is a stratification of ∂M into Nash manifolds of dimension k, that $X \cap \partial M$ is the union of some connected components of $M_0, ..., M_{n-1}$, and the method of construction of $\{M_k\}$ is **canonical**.

Lemma 4.6. *Let M be a non-compact Nash manifold contained and closed in $\mathbf{R}^N$ and X a normal crossing Nash subset of M. Let $B(r)$ denote the closed ball in $\mathbf{R}^N$ with center 0 and radius $r \in \mathbf{R}$. Then there exists a Nash diffeomorphism $\tau : M \to M \cap \operatorname{Int} B(r)$, for some large r, such that $\tau(X) = X \cap \operatorname{Int} B(r)$.*

This does not necessarily hold in the analytic case.

Proof of lemma 4.6. Assume that M is of dimension n. Set $X_n = M - X$. Choose r so large that the $p|_{X_i - B(r/2)}$ are submersions onto $(r/2, \infty)$, where $\{X_i : i = 0, ..., n-1\}$ denotes the canonical stratification of X and $p(x) = |x|$ for $x \in M$. Then by lemma 4.5 there exists a Nash diffeomorphism $\rho : M - B(r/2) \to (B \cap p^{-1}(r)) \times (r/2, \infty)$ of the form $\rho = (\rho', p)$ such that $\rho'(X - B(r/2)) = X \cap p^{-1}(r)$. Let $\alpha : (-\infty, r) \to \mathbf{R}$ be a semialgebraic C^l diffeomorphism such that $\alpha = \mathrm{id}$ on $(-\infty, r/2)$, where l is a sufficiently large integer. Set

$$\tau_0(x) = \begin{cases} x & \text{for } x \in M \cap B(r/2) \\ \rho^{-1}(\rho'(x), \alpha^{-1} \circ p(x)) & \text{for } x \in M - B(r/2). \end{cases}$$

Then τ_0 is a semialgebraic C^l diffeomorphism from M to $M \cap \operatorname{Int} B(r)$ such that $\tau_0(X) = X \cap \operatorname{Int} B(r)$. We only need to approximate τ_0 by a Nash diffeomorphism keeping the last property. Let $\pi : M \to M \cap \operatorname{Int} B(r)$ be a Nash approximation of τ_0 in the semialgebraic C^l topology. Replace τ_0 with $\pi \circ \tau_0^{-1}$. Then what we prove is the following statement.

Let $\tilde{M}$ be a compact Nash manifold with boundary in $\mathbf{R}^N$, let $\tilde{X}$ be a normal crossing Nash subset of $\tilde{M}$ with $\partial \tilde{M} \not\subset \tilde{X}$, and let $\tilde{\tau}_0$ be a semialgebraic C^l diffeomorphism of $\operatorname{Int} \tilde{M}$ arbitrarily close to id in the semialgebraic C^l topology such that $\tilde{\tau}_0(\tilde{X} \cap \operatorname{Int} \tilde{M})$ is a normal crossing Nash subset of $\operatorname{Int} \tilde{M}$. Then we can approximate $\tilde{\tau}_0$ by a Nash diffeomorphism $\tilde{\tau}$ of $\operatorname{Int} \tilde{M}$ in the semialgebraic C^1 topology so that $\tilde{\tau}(\tilde{X} \cap \operatorname{Int} \tilde{M}) = \tilde{\tau}_0(\tilde{X} \cap \operatorname{Int} \tilde{M})$.

We proceed as in the proof of step 1, theorem 3.1,(1). Let $\{\tilde{X}_j : j = 0, ..., n-1\}$ denote the canonical stratification of $\tilde{X}$ and set $\tilde{X}_n = \tilde{M} - X$. By induction, for some $i \in \mathbf{N}$, assume that $\tilde{\tau}_0|_{\cup_{j=0}^{i-1} \tilde{X}_j \cap \operatorname{Int} \tilde{M}}$ is of class Nash. Let $l' \in \mathbf{N}$. Then it suffices to choose l large enough and to approximate $\tilde{\tau}_0$ by a semialgebraic C^l diffeomorphism $\tilde{\tau}$ of $\operatorname{Int} \tilde{M}$ in the semialgebraic $C^{l'}$ topology so that $\tilde{\tau}(\tilde{X} \cap \operatorname{Int} \tilde{M}) = \tilde{\tau}_0(\tilde{X} \cap \operatorname{Int} \tilde{M})$ and $\tilde{\tau}|_{\cup_{j=0}^{i} \tilde{X}_j \cap \operatorname{Int} \tilde{M}}$ is of class Nash. Let $\mathcal{I}$ denote the sheaf of $\mathcal{N}$-ideals on $\operatorname{Int} \tilde{M}$ defined by $\cup_{j=0}^{i-1} \tilde{X}_j \cap \operatorname{Int} \tilde{M}$. By theorem 2.7, the sheaf $\mathcal{I}$ is generated by a finite number of global cross-sections $\xi_1, ..., \xi_k$ of $\mathcal{I}$. Then $\tilde{\tau}_0|_{\cup_{j=0}^{i-1} \tilde{X}_j \cap \operatorname{Int} \tilde{M}}$ is

Hence by theorem 2.8 we have a Nash map $h : \operatorname{Int}\tilde{M} \to \mathbf{R}^N$ such that $h = \tilde{\tau}_0$ on $\cup_{j=0}^{i-1}\tilde{X}_j \cap \operatorname{Int}\tilde{M}$. Here we can choose h to be sufficiently close to $\tilde{\tau}_0$ in the semialgebraic $C^{l'}$ topology for the following reason. It suffices to see that $\tilde{\tau}_0 - h$ is of the form $\sum_{j=1}^k \xi_j\beta_j$ for some semialgebraic $C^{l'}$ maps $\beta_j : \operatorname{Int}\tilde{M} \to \mathbf{R}^N$ because $h + \sum_{j=1}^k \xi_j\tilde{\beta}_j$ fulfills the requirements, where $\tilde{\beta}_j$ denote Nash approximations of β_j in the semialgebraic $C^{l'}$ topology. Hence we will prove the following statement.

Let β be a semialgebraic C^l function on $\operatorname{Int}\tilde{M}$ vanishing on $\cup_{j=0}^{i-1}\tilde{X}_j \cap \operatorname{Int}\tilde{M}$. Then β is of the form $\sum_{j=1}^k \beta_j\xi_j$ for some semialgebraic $C^{l'}$ functions β_j on $\operatorname{Int}\tilde{M}$.

By the second induction, assume that the statement holds for manifolds of dimension strictly less than n. The problem is reduced to the Euclidean case as follows. There exists a finite open semialgebraic covering $\{O_s\}$ of $\operatorname{Int}\tilde{M}$ such that each $(O_s, O_s \cap \tilde{X})$ is Nash diffeomorphic to $(\mathbf{R}^n, \{(x_1, ..., x_n) \in \mathbf{R}^n : x_1 \cdots x_{n_s} = 0\})$ for some $n_s \in \mathbf{N}$. Let $\{\eta_s\}$ and $\{\eta'_s\}$ be a partition of unity of class semialgebraic C^l subordinate to $\{O_s\}$, and semialgebraic C^l functions on $\operatorname{Int}\tilde{M}$, respectively, such that $\eta'_s = 1$ on $\operatorname{supp}\eta_s$ and $\operatorname{supp}\eta'_s \subset O_s$. If each $(\beta\eta_s)|_{O_s}$ is described to be of the form $\sum_j \beta_{j,s}\xi_j|_{O_s}$ for some semialgebraic $C^{l'}$ functions $\beta_{j,s}$ on O_s then the naturally defined functions $\sum_s \beta_{j,s}\eta'_s$, for $j = 1, ..., k$, are semialgebraic $C^{l'}$ functions on $\operatorname{Int}\tilde{M}$ and $\beta = \sum_j(\sum_s \beta_{j,s}\eta'_s)\xi_j$. Hence we can assume that $(\operatorname{Int}\tilde{M}, \operatorname{Int}\tilde{M} \cap X) = (\mathbf{R}^n, \{x_1 \cdots x_{n'} = 0\})$ for some $n' \in \mathbf{N}$, and then $n' > 0$. Apply the induction hypothesis to $\beta|_{\{x_1=0\}}$. Then there exist semialgebraic C^{l_1} functions β'_j on $\mathbf{R}^{n-1}$ such that

$$\beta(0, x_2, ..., x_n) = \sum_{j=1}^k \beta'_j(x_2, ..., x_n)\xi_j(0, x_2, ..., x_n)$$

because $\mathcal{I}|_{\{x_1=0\}}$ is the sheaf of $\mathcal{N}$-ideals on $\{x_1 = 0\}$ defined by $\cup_{j=0}^{i-1}\tilde{X}_j \cap \{x_1 = 0\}$ (here $l_1 > 0$ is arbitrarily given and l depends on l_1). Regard naturally β'_j as semialgebraic C^{l_1} functions on $\mathbf{R}^n$ and replace β with $\beta - \sum \beta'_j\xi_j$. Then we can suppose that $\beta = 0$ on $\{x_1 = 0\}$ from the beginning. Under this assumption β/x_1 is a well-defined semialgebraic C^{l_1-1} function. Consider β/x_1 and $\{x_2 \cdots x_{n'} = 0\}$ in place of β and $\{x_1 \cdots x_{n'} = 0\}$, and repeat the same arguments for $\{x_2 = 0\}$ and so on. Then we finally arrive at the case $\tilde{X} = \emptyset$. Thus the statement is proved, and h is chosen to be close to $\tilde{\tau}_0$ in the semialgebraic $C^{l'}$ topology.

Set $Y = \tilde{\tau}_0(\tilde{X} \cap \operatorname{Int}\tilde{M})$ and $Y_j = \tilde{\tau}_0(\tilde{X}_j \cap \operatorname{Int}\tilde{M})$. Then Y is a normal crossing Nash subset of $\operatorname{Int}\tilde{M}$, the set $\{Y_j : j = 0, ..., n-1\}$ is its canonical stratification, and $\overline{Y}$ is a normal crossing semialgebraic C^l subset of $\tilde{M}$ in the sense that $\tilde{M}$ has a semialgebraic C^l local coordinate system $(x_1, ..., x_n)$ at each point of $\partial\tilde{M}$ with $\overline{Y} = \{x_1 \geq 0,\ x_2 \cdots x_{n'} = 0\}$ for some $n' > 0 \in \mathbf{N}$ by the definition of semialgebraic C^l topology. Hence there exists a tubular neighborhood U_i of Y_i in $\mathbf{R}^N$ such that for some $\epsilon > 0 \in \mathbf{R}$

$$U_i = \cup_{y\in Y_i}\{x \in \mathbf{R}^N : |x - y| < \epsilon\,\mathrm{dis}(y, \cup_{j=0}^{i-1}Y_j),\ (x - y) \perp T_yY_i\}.$$

Let $q_i : U_i \to Y_i$ denote the orthogonal projection. Choose h so close to $\tilde{\tau}_0$ that $h(\tilde{X}_i \cap \operatorname{Int}\tilde{M}) \subset U_i$. Then $q_i \circ h|$ [illegible] is a Nash map to $\cup_{j=0}^{i}Y_j$ close to

$\tilde\tau_0|_{\cup_{j=0}^i \tilde X_j \cap \operatorname{Int} \tilde M}$ in the semialgebraic $C^{l'}$ topology. Note that the map is a diffeomorphism by Lemma II.1.7 in [S$_2$]. Hence it remains only to extend it to a semialgebraic C^l approximation $\tilde\tau : \operatorname{Int} \tilde M \to \operatorname{Int} \tilde M$ of $\tilde\tau_0$ in the semialgebraic $C^{l'}$ topology so that $\tilde\tau(\tilde X \cap \operatorname{Int} \tilde M) = Y$. However, we have already proved it without the last condition. Moreover, the proof shows also that the condition is furnished inductively. Thus we complete the construction of τ. $\square$

Lemma 4.7. *Let f and g be Nash functions on a Nash manifold M which have the same sign at each point of M, only normal crossing singularities at the common zero set X and the same multiplicity at each point of X. Let $l \in \mathbf{N}$. Then there exists a Nash diffeomorphism π of M such that $\pi(X) = X$ and $f - g\circ\pi$ is l-flat at X.*

If f is fixed and g is chosen such that the Nash function on M, defined to be g/f on $M - X$, is close to 1 in the Nash topology, then π is chosen to be close to id in the Nash topology.

Proof of lemma 4.7. Let $M \subset \mathbf{R}^N$, set $n = \dim M$ and let l be sufficiently large. For each $k\,(< n) \in \mathbf{N}$, let X_k denote the union of the strata of the canonical stratification of X of dimension less than or equal to k. By induction, assume that $f - g$ is l-flat at X_{k-1} for some k. Then we need only to find a Nash diffeomorphism π of M such that $\pi - \mathrm{id}$ is l-flat at X_{k-1}, such that $\pi(X) = X$ and $f - g\circ\pi$ is l-flat at X_k (to be precise, we will construct π so that $\pi - \mathrm{id}$ and $f - g\circ\pi$ are $l^{(4)}$-flat at X_{k-1} and X_k, respectively, for some $0 \ll l^{(4)} \ll \cdot\cdot \ll l' \ll l$).

We proceed as in the proof of lemma 4.5. Let $(\tilde M, \tilde X)$ and $(\tilde M_k, \tilde X_k)$ be pairs of Nash manifolds and Nash submanifolds, let $p : \tilde M \to M$ and $p_k : \tilde M_k \to M$ be Nash immersions and let $q_k : \tilde M_k \to \tilde X_k$ be a Nash submersive retraction such that $\dim \tilde M = \dim \tilde M_k = n$, the equalities $p(\tilde X) = X$ and $p_k(\tilde X_k) = X_k$ hold, and moreover $p|_{\tilde X - p^{-1}(X_{n-2})}$ and $p_k|_{\tilde X_k - p_k^{-1}(X_{k-1})}$ are injective, and $p_k(q_k^{-1}(\tilde X_k \cap p_k^{-1}(X_{k-1}))) \subset X$. Shrink $\tilde M_k$ if necessary. Then we have an open semialgebraic neighborhood U of $\tilde X \cap p^{-1}(X_k)$ in $\tilde M$ and a Nash $(n-k)$-fold covering map $r : U \to \tilde M_k$ such that $p_k \circ r = p$ on U. Let $\tilde\phi$ be a Nash function on $\tilde M$ with zero set $\tilde X$ which is, locally at each point of $\tilde X$, the square of a regular function. Then $\tilde\phi(r^{-1}(x))$ is a family of $(n-k)$-numbers possibly with multiplicity, for each $x \in \tilde M_k$, and there exist Nash functions $\tilde\phi_{k,1}, ..., \tilde\phi_{k,n-k}$ on an open semialgebraic neighborhood of each point of $\tilde M_k$ such that $\tilde\phi(r^{-1}(x)) = \{\tilde\phi_{k,1}(x), ..., \tilde\phi_{k,n-k}(x)\}$ for x in the given neighborhood. For simplicity of notation, we assume that $\tilde\phi_{k,1}(x), ..., \tilde\phi_{k,n-k}(x)$ are defined globally, which causes no problem because the following arguments are done locally and do not depend on the order of $\tilde\phi_{k,1}(x), ..., \tilde\phi_{k,n-k}(x)$. Moreover, we suppose that each $\tilde\phi_{k,i}$ is the square of a regular Nash function, say $\tilde\phi_{k,i}^{1/2}$, by the same reason as above. Set $\tilde f_k = f\circ p_k$ and $\tilde g_k = g \circ p_k$.

We want to construct a Nash diffeomorphism $\tilde\pi_k$ between semialgebraic neighborhoods of $\tilde X_k$ in $\tilde M_k$ such that $\tilde\pi_k(p_k^{-1}(X)) \subset p_k^{-1}(X)$, such that $\tilde\pi_k - \mathrm{id}$ is l''-flat at $q_k^{-1}(\tilde X_k \cap p_k^{-1}(X_{k-1}))$ and $\tilde f_k - \tilde g_k \circ \tilde\pi_k$ is l''-flat at $\tilde X_k$. Assume that $\tilde X_k$ is connected without loss of generality. Since $\tilde f_k$ and $\tilde g_k$ have only normal crossing singularities at $p_k^{-1}(X)$, the same sign at each point of $\tilde M_k$ and the same multiplicity at each point of $p_k^{-1}(X)$, and since $\tilde f_k^{-1}(0) = \tilde g_k^{-1}(0) = \cup_{i=1}^{n-k} (\tilde\phi_{k,i}^{1/2})^{-1}(0) \cup q_k^{-1}(\tilde X_k \cap p_k^{-1}(X_{k-1}))$,
we have Nash functions F and G on $\tilde M_k$ and

such that the equalities $\tilde f_k = F\tilde\phi_k^{1/2\alpha}$ and $\tilde g_k = G\tilde\phi_k^{1/2\alpha}$ hold, such that $FG \ge 0$ on $\tilde M_k$ and $FG > 0$ on $\tilde M_k - q_k^{-1}(\tilde X_k \cap p_k^{-1}(X_{k-1}))$, where $\tilde\phi_k^{1/2\alpha} = \prod_{i=1}^{n-k} \tilde\phi_{k,i}^{1/2\alpha_i}$. Assume that $F \ge 0$ and hence $G \ge 0$ (the other cases can be proved in the same way). Note that F and G have zero set $q_k^{-1}(\tilde X_k \cap p_k^{-1}(X_{k-1}))$, which has only normal crossing singularities and has the same multiplicity at each point. Shrink $\tilde M_k$ so that the map $(q_k, \tilde\phi_{k,1}^{1/2}, ..., \tilde\phi_{k,n-k}^{1/2}) : \tilde M_k \to \tilde X_k \times \mathbf{R}^{n-k}$ is a Nash embedding and let V denote its image. Identify $\tilde M_k$ and $\tilde X_k$ with V and $\tilde X_k \times \{0\}$ through this embedding, set $\tilde p_k = p_k|_{\tilde X_k}$, regard p_k as an immersion of V into M and $\tilde f_k$ and $\tilde g_k$ as functions on V, and let $(z, y) = (z, y_1, ..., y_{n-k}) \in V \subset \tilde X_k \times \mathbf{R}^{n-k}$. Then

$$\tilde f_k(z, y) = F(z, y)y^\alpha \quad \text{and} \quad \tilde g_k(z, y) = G(z, y)y^\alpha.$$

Set

$$F' = \sum_{\beta \in \mathbf{N}_l^{n-k}} \frac{\partial^{|\beta|}F}{\partial y^\beta}(z, 0)y^\beta/\beta!, \quad G' = \sum_{\beta \in \mathbf{N}_l^{n-k}} \frac{\partial^{|\beta|}G}{\partial y^\beta}(z, 0)y^\beta/\beta!,$$
$$\tilde f_k' = F'y^\alpha \quad \text{and} \quad \tilde g_k' = G'y^\alpha,$$

where $\mathbf{N}_l^{n-k} = \{\beta \in \mathbf{N}^{n-k} : |\beta| \le l\}$ and $\beta! = \prod_{i=1}^{n-k} \beta_i!$. Then $\tilde f_k'$ and $\tilde g_k'$ are Nash functions on V, moreover $\tilde f_k - \tilde f_k'$ and $\tilde g_k - \tilde g_k'$ are l-flat at $\tilde X_k \times \{0\}$, and F' and G' have the same properties as F and G. Hence for the construction of $\tilde\pi_k$, we can replace $\tilde f_k$ and $\tilde g_k$ with $\tilde f_k'$ and $\tilde g_k'$. An advantage of $\tilde f_k'$ and $\tilde g_k'$ is the fact that $(*)$ $F' - G'$ is l'-flat at $V \cap \tilde p_k^{-1}(X_{k-1}) \times \mathbf{R}^{n-k}$, though $F - G$ is l'-flat only at $\tilde p_k^{-1}(X_{k-1}) \times \{0\}$. Write

$$\tilde f_k' = \prod_{i=1}^{n-k} (F'^{1/|\alpha|}y_i)^{\alpha_i} \quad \text{and} \quad \tilde g_k' = \prod_{i=1}^{n-k} (G'^{1/|\alpha|}y_i)^{\alpha_i}.$$

Then there exists a unique Nash diffeomorphism $\tilde\pi_k$ between semialgebraic neighborhoods of $\tilde X_k \times \{0\}$ in V of the form $\tilde\pi_k(z, y) = (z, \tilde\pi_k'(z, y)y)$, for some positive Nash function $\tilde\pi_k'$ on the neighborhood of source, such that $\tilde f_k' = \tilde g_k' \circ \tilde\pi_k$ on that neighborhood. Actually, we can reduce the problem to the case where $\tilde g_k' = z^\beta y^\alpha$ for some $\beta \in \mathbf{N}^k$ and some local Nash coordinate system $z = (z_1, ..., z_k)$ of $\tilde X_k$ such that $\tilde p_k^{-1}(X_{k-1}) = \{z^\beta = 0\}$ (by considering two pairs $(\tilde f_k', z^\beta y^\alpha)$ and $(\tilde g_k', z^\beta y^\alpha)$) and then $\tilde\pi_k'(z, y) = (F'/z^\beta)^{1/|\alpha|}$ is the unique solution. Such a $\tilde\pi_k$ fulfills the requirements. Indeed, $\tilde\pi_k(p_k^{-1}(X)) \subset p_k^{-1}(X)$ by the form of $\tilde\pi_k$ because $p_k^{-1}(X)$ in V is of the form $\tilde X_k \times \{y_1 \cdots y_{n-k} = 0\} \cup \tilde p_k^{-1}(X_{k-1}) \times \mathbf{R}^{n-k}$, because $\tilde\pi_k - \mathrm{id}$ is l''-flat at $V \cap \tilde p_k^{-1}(X_{k-1}) \times \mathbf{R}^{n-k}$ because of $(*)$, and $\tilde f_k - \tilde g_k \circ \tilde\pi_k$ is l''-flat at $\tilde X_k \times \{0\}$ because

$$\tilde f_k - \tilde g_k \circ \tilde\pi_k = (\tilde f_k - \tilde f_k') + (\tilde f_k' - \tilde g_k' \circ \tilde\pi_k) + (\tilde g_k' \circ \tilde\pi_k - \tilde g_k \circ \tilde\pi_k).$$

Let W be an open semialgebraic neighborhood of $X_k - X_{k-1}$ in M so small that there exists an open semialgebraic neighborhood of $(\tilde X_k - \tilde p_k^{-1}(X_{k-1})) \times \{0\}$ in the intersection of the domain of definition of $\tilde\pi_k$ and the range of values to which the restriction of p_k is a diffeomorphism onto W. Then $\tilde\pi_k$ induces a Nash embedding

$\pi_k : W \to M \subset \mathbf{R}^n$ such that $\pi_k(X \cap W) \subset X$, such that $\pi_k = \mathrm{id}$ on $X_k - X_{k-1}$ and $f - g \circ \pi_k$ is l''-flat at $X_k - X_{k-1}$. Though π_k is not necessarily extensible to a neighborhood of X_k, there exists a Nash map $\eta : M \to \mathbf{R}^N$ such that $\eta - \mathrm{id}$ is l''-flat at X_{k-1}, and $\eta - \pi_k$ is l''-flat at $X_k - X_{k-1}$, hence $f - g \circ \eta$ is l''-flat at X_k for the following reason. Let $\mathcal{I}$ denote the sheaf of $\mathcal{N}$-ideals on M defined by X_k. Then by theorem 2.8, it suffices to find an element $\overline{\eta}$ in $H^0(M, \mathcal{N}/\mathcal{I}^{l''})^N$ such that $\overline{\eta}_x$ is the image of π_{kx} under the natural map $\mathcal{N}_x^n \to (\mathcal{N}_x/\mathcal{I}_x^{l''})^N$ for $x \in W$ and $\overline{\eta}_x = \mathrm{id}$ for $x \in X_{k-1}$. This is possible because $\tilde{\pi}_k - \mathrm{id}$ is l''-flat at $V \cap \tilde{p}_k^{-1}(X_k) \times \mathbf{R}^{n-k}$.

We modify $\tilde{\pi}_k$ to show that η can be a diffeomorphism of M. Assume that $(**)$ $\tilde{\pi}'_k \le 1$ for simplicity of notation, which is possible if we consider a third function h on M with the same properties as f and g, with $h/f \ge 1$ on $M - X$ and $h/g \ge 1$ on $M - X$. Let ψ be a non-negative small Nash function on $\tilde{X}_k$ with zero set $\tilde{p}_k^{-1}(X_{k-1})$ such that

$$
\begin{aligned}
&Z \overset{\mathrm{def}}{=} \{(z, y) \in \tilde{X}_k \times \mathbf{R}^{n-k} : |y| \le \psi(z)\} \subset \text{domain of } \tilde{\pi}_k,\\
(3*)\qquad &\tilde{\pi}'_k(z, sy) > \left|\frac{\partial \tilde{\pi}'_k(z, sy)}{\partial s} s\right|\\
&\qquad \text{for } (z, y, s) \in \tilde{X}_k \times \mathbf{R}^{n-k} \times \mathbf{R} \text{ with } (z, sy) \in Z \text{ and } |y| = 1
\end{aligned}
$$

and $p_k|_Z$ is injective, which exists by the Łojasiewicz inequality. Let $\rho(t)$ be a semialgebraic $C^{l''}$ function on $\mathbf{R}$ such that $(4*)$ $0 \le \rho \le 1$, such that $(5*)$ $\frac{d\rho}{dt} \le 0$, and moreover $\rho = 1$ on $(-\infty, 1/2]$ and $\rho = 0$ on $[1, \infty)$. Set

$$
\tilde{\tau}'_k = \begin{cases} 1 & \text{for } (z, y) \in Z \cap p_k^{-1}(X_{k-1}) \\ \rho(|y|/\psi(z))\tilde{\pi}'_k(z, y) + 1 - \rho(|y|/\psi(z)) & \text{for } (z, y) \in Z - p_k^{-1}(X_{k-1}), \end{cases}
$$
$$
\tilde{\tau}_k(z, y) = (z, \tilde{\tau}'_k(z, y)y) \quad \text{for } (z, y) \in Z.
$$

Then $\tilde{\tau}'_k$ and hence $\tilde{\tau}_k$ are of class semialgebraic $C^{l^{(3)}}$ and $\tilde{\tau}_k - \mathrm{id}$ is $l^{(3)}$-flat at $Z \cap p_k^{-1}(X_{k-1}) = \tilde{p}_k^{-1}(X_{k-1}) \times \{0\}$ since $\tilde{\pi}'_k(z, y) - 1$ is $(l'' - 1)$-flat at $Z \cap p_k^{-1}(X_{k-1})$. Clearly $\tilde{\tau}_k = \mathrm{id}$ on a semialgebraic neighborhood of $\partial Z - p_k^{-1}(X_{k-1})$ in Z. Moreover, $\tilde{\tau}_k$ is a diffeomorphism of Z. Actually, we can assume that $n - k = 1$ because $\tilde{\tau}_k = \tilde{\pi}_k$ on a neighborhood of $(\tilde{X}_k - \tilde{p}_k^{-1}(X_{k-1})) \times \{0\}$ in Z and because $\tilde{\pi}_k$ and hence $\tilde{\tau}_k$ carry each segment $\{z\} \times \{\mathbf{R}y\} \cap Z$ for $(z, y) \in (\tilde{X}_k - \tilde{p}_k^{-1}(X_{k-1})) \times \mathbf{R}^{n-k}$ with $|y| = 1$ to itself. Then

$$
\frac{\partial \tilde{\tau}'_k(z, y)y}{\partial y} = \tilde{\tau}'_k(z, y) + \frac{\partial \tilde{\tau}'_k}{\partial y}(z, y)y,
$$

$$
\tilde{\tau}'_k(z, y) = \rho(|y|/\psi(z))\tilde{\pi}'_k(z, y) + 1 - \rho(|y|/\psi(z)) \overset{(**),(4*)}{\ge} \tilde{\pi}'_k(z, y),
$$

$$
\frac{\partial \tilde{\tau}'_k}{\partial y}(z, y)y = \frac{d\rho}{dt}(|y|/\psi(z))(\tilde{\pi}'_k(z, y) - 1)|y|/\psi(z) + \rho(|y|/\psi(z))\frac{\partial \tilde{\pi}'_k}{\partial y}(z, y)y \overset{(**),(5*)}{\ge}
$$
$$
\rho(|y|/\psi(z))\frac{\partial \tilde{\pi}'_k}{\partial y}(z, y)y, \quad \text{hence}
$$

$$
\frac{\partial \tilde{\tau}'_k(z, y)y}{\partial y} \ge \tilde{\pi}'_k(z, y) + \rho(|y|/\psi(z))\frac{\partial \tilde{\pi}'_k}{\partial y}(z, y)y \overset{(3*),(4*)}{>} 0 \quad \text{for } (z, y) \in Z.
$$

Define a semialgebraic $C^{l^{(3)}}$ diffeomorphism τ_k of M so that $\tau_k \circ p_k = p_k \circ \tilde{\tau}_k$ on Z and $\tau_k = \mathrm{id}$ on $M - p_k(Z)$. Then $\tau_k = \pi_k$ on Z if we shrink Z, hence $\tau_k - \eta$ is $l^{(3)}$-flat at X_k and $\tau_k(X) = X$ and moreover $f - g \circ \tau_k$ is $l^{(3)}$-flat at X_k. Let ω be a non-negative-valued global generator of the square of $\mathcal{I}$—the sheaf of $\mathcal{N}$-ideals defined by X_k. Then there exists a semialgebraic $C^{l^{(4)}}$ map $\xi : M \to \mathbf{R}^N$ such that $\tau_k - \eta = \omega\xi$. Approximate ξ by a Nash map ξ' in the semialgebraic $C^{l^{(4)}}$ topology, and set $\pi = (\eta + \omega\xi') \circ o$, where o denotes the orthogonal projection to M of its semialgebraic tubular neighborhood in $\mathbf{R}^N$. Then π is a Nash diffeomorphism of M such that $\pi - \mathrm{id}$ is $l^{(4)}$-flat at X_{k-1} and $f - g \circ \pi$ is $l^{(4)}$-flat at X_k. We can modify π so that $\pi(X) = X$ in the same way as in step 1 of the proof of theorem 3.1,(1) and lemma 4.6, because π is an approximation of τ_k and $\tau_k(X) = X$. Thus we complete the proof of the former half of lemma 4.7.

The latter half automatically follows from the above proof (though $(**)$ does not necessarily hold, π'_k is close to 1 in the Nash topology, which is sufficient to proceed). □

Note that our proof of lemma 4.7 also works when M, f and g are of class C^ω and the multiplicities of f and g are bounded.

The following lemma is also a globalization of Chapter II, Proposition 2 in [T] and shows sufficient conditions for two functions to be right equivalent.

Proposition 4.8. *(i) Let f be a C^ω function on a C^ω manifold M. Let v_i, for $i = 1, ..., k$, be C^ω vector fields on M, and I denote the ideal of $C^\infty(M)$ or $C^\omega(M)$ generated by $v_i f$, for $i = 1, ..., k$. Let ϕ be a small C^∞ or C^ω function on M contained in I^2 in the strong Whitney C^∞ topology. Then f and $f + \phi$ are C^∞ or C^ω right equivalent, respectively, and the diffeomorphism of equivalence can be chosen to be close to id in the same topology.*

(ii) If f, M and v_i are of class Nash or C^∞ or C^ω, and ϕ is of the form $\sum_{i,j=1}^k \phi_{i,j} v_i f \cdot v_j f$ for some small Nash or C^∞ or C^ω functions $\phi_{i,j}$ in the Nash or (strong) Whitney C^∞ topology, then f and $f + \phi$ are Nash or C^∞ or C^ω right equivalent, respectively, by a Nash or C^∞ or C^ω diffeomorphism close to id in the same topology.

(iii) Assume that M is a Nash manifold and f is a Nash function on M with only normal crossing singularities. Set $X = f^{-1}(f(\mathrm{Sing}\, f))$. Let ϕ be a Nash function on M r-flat at X for some large $r \in \mathbf{N}$. Then there exists a Nash diffeomorphism $\pi : V_1 \to V_2$ between closed semialgebraic neighborhoods of X in M close to id in the semialgebraic $C^{r'}$ topology, for $0 < r' (\ll r) \in \mathbf{N}$, such that $f \circ \pi = f + \phi$ on V_1, such that $\pi - \mathrm{id}$ is r'-flat at X, and π is extensible to a semialgebraic C^r diffeomorphism of M.

Proof of proposition 4.8. Consider the analytic case. We want to reduce (i) to (ii). For a while we proceed in the strong Whitney C^∞ topology. By lemma 1.12 for ϕ in (i), there exist small $\phi_{i,j} \in C^\omega(\mathrm{M})$, for $i, j = 1, ..., k$, such that $\phi = \sum_{i,j=1}^k \phi_{i,j} v_i f \cdot v_j f$. Consequently, (i) is reduced to (ii). From now on, we work in the Whitney C^r topology for any $r > 0 \in \mathbf{N}$ (though we can do in the strong Whitney C^∞ topology). We can assume that M is open in its ambient Euclidean. Actually, let $p : \tilde{M} \to M$ denote the orthogonal projection of a tubular neighborhood of M in its ambient Euclidean space. Assume that proposition 4.8,(ii) in the analytic case holds for $\tilde{M}$. The map $C^\omega(\tilde{M}) \ni \Psi \to \Psi|_M \in C^\omega(M)$ is obviously continuous, surjective by corollary 2.4 and open as follows. Let $\xi \in$

$C^\infty(\tilde{M})$ with $\xi = 1$ on M and $\xi = 0$ outside of a small neighborhood of M in $\tilde{M}$. Then the map $\xi C^\omega(\tilde{M}) \ni \xi\Psi \to \xi\Psi|_M \in C^\omega(M)$ is open because for $\psi \in C^\omega(M)$ and for $\Psi_0 \in C^\infty(\tilde{M})$, we have $\psi \circ p \in C^\omega(\tilde{M})$ and $(\xi \cdot \psi \circ p)|_M = \psi$ and the map $C^\omega(M) \ni \psi \to \xi \cdot \psi \circ p + \xi\Psi_0 - \xi \cdot \Psi_0|_M \circ p \in \xi C^\omega(\tilde{M})$ is continuous and carries $\Psi_0|_M$ to $\xi\Psi_0$. Hence for small $\psi \in C^\omega(M)$, there exists small $\xi\Psi \in \xi C^\omega(\tilde{M})$ such that $\Psi|_M = \psi$. Approximate ξ by an analytic function ξ' on $\tilde{M}$ so that $\xi' = 1$ on M. Then $\xi'\Psi$ is analytic on $\tilde{M}$, close to $\xi\Psi$ and hence small since the map $C^\infty(\tilde{M})^2 \ni (\alpha, \beta) \to \alpha\beta \in C^\infty(\tilde{M})$ is continuous, and $\xi'\Psi|_M = \psi$. Consequently, the above restriction map $\Psi \to \Psi|_M$ is open by linearity. Let $\tilde{v}_i$, for $i = 1, ..., k$, be C^ω vector field extensions of v_i to $\tilde{M}$, and $\tilde{\phi}_{i,j}$ C^ω extensions of $\phi_{i,j}$ to $\tilde{M}$ so small that $f \circ p$ and $f \circ p + \sum_{i,j}^k \tilde{\phi}_{i,j}\tilde{v}_i(f \circ p) \cdot \tilde{v}_j(f \circ p)$ satisfy the condition in proposition 4.8,(ii) and hence are C^ω right equivalent by a C^ω diffeomorphism $\tilde{\pi}$ close to id, i.e.,

$$f \circ p \circ \tilde{\pi} = f \circ p + \sum_{i,j=1}^{k} \tilde{\phi}_{i,j}\tilde{v}_i(f \circ p) \cdot \tilde{v}_j(f \circ p) \quad \text{on } \tilde{M}.$$

Set $\pi = p \circ \tilde{\pi}|_M$. Then π is a C^ω diffeomorphism of M close to id, and

$$f \circ \pi = f + \sum_{i,j=1}^{k} \phi_{i,j} v_i f \cdot v_j f.$$

Thus proposition 4.8,(ii) is proved for M. Hence we assume that M is open in $\mathbf{R}^n$.

Next we can suppose that $k = n$ and $v_i = \frac{\partial}{\partial x_j}$, for $i = 1, ..., n$, because each v_i is written as $\sum_{j=1}^n \alpha_{i,j} \frac{\partial}{\partial x_j}$ for some C^ω functions $\alpha_{i,j}$ on M.

Let η denote the function on M which measures distance from $\partial M \overset{\text{def}}{=} \overline{M} - M$ (if $\partial M = \emptyset$ then set $\eta \equiv +\infty$). Set $V = \{(x, y) \in M \times \mathbf{R}^n : |y| < \eta(x)\}$ and consider the C^ω function

$$g(x, y) = f(x + y) - f(x) - \sum_{i=1}^{n} y_i \frac{\partial f}{\partial x_i}(x) \quad \text{for } (x, y) = (x_1, ..., x_n, y_1, ..., y_n) \in V.$$

Then g is a global cross-section of the sheaf of $\mathcal{O}$-ideals $\mathcal{I}$ on V generated by $y_i y_j$, for $i, j = 1, ..., n$. Hence applying theorem 2.3 to the surjective homomorphism $\mathcal{O}^{n^2} \ni (\alpha_{i,j}) \to \sum_{i,j=1}^n \alpha_{i,j} y_i y_j \in \mathcal{I}$ we obtain C^ω functions $g_{i,j}$ on V, $i, j = 1, ..., n$, such that $g(x, y) = \sum_{i,j=1}^n y_i y_j g_{i,j}(x, y)$. Then

$$(*) \qquad f(x + y) = f(x) + \sum_{i=1}^{n} y_i \frac{\partial f}{\partial x_i}(x) + \sum_{i,j=1}^{n} y_i y_j g_{i,j}(x, y).$$

Let $\alpha = (\alpha_{i,j})_{i,j=1,...,n}$ be new variables in $\mathbf{R}^{n^2}$, set

$$\langle \alpha, \partial f \rangle = (\sum_{i=1}^{n} \alpha_{i,1} \frac{\partial f}{\partial x_i}(x), ..., \sum_{i=1}^{n} \alpha_{i,n} \frac{\partial f}{\partial x_i}(x)),$$

and let W be a small open neighborhood of $M \times \{0\}$ in $M \times \mathbf{R}^{n^2}$ such that

$$(x, \langle \alpha, \partial f \rangle) \in V \quad \text{for } (x, \alpha) \in W.$$

Take y to be $\langle \alpha, \partial f \rangle$ in $(*)$. Then

$$f(x + \langle \alpha, \partial f \rangle) =$$
$$f(x) + \sum_{i,j}^{n} \alpha_{i,j} \frac{\partial f}{\partial x_i}(x) \frac{\partial f}{\partial x_j}(x) + \sum_{i,i',j,j'=1}^{n} \alpha_{i,i'} \alpha_{j,j'} \frac{\partial f}{\partial x_i}(x) \frac{\partial f}{\partial x_j}(x) G_{i',j'}(x, \alpha)$$

for C^ω functions $G_{i',j'}(x, \alpha) = g_{i',j'}(x, \langle \alpha, \partial f \rangle)$ on W. Consider the map

$$B : W \ni (x, \alpha) \to (x, \alpha_{i,j} + \sum_{i',j'=1}^{n} \alpha_{i,i'} \alpha_{j,j'} G_{i',j'}(x, \alpha)) \in M \times \mathbf{R}^{n^2}.$$

Then B is id and regular at $M \times \{0\}$. Hence, shrinking W, we assume that B is a diffeomorphism onto an open neighborhood O of $M \times \{0\}$ in $M \times \mathbf{R}^{n^2}$. Set $B(x, \alpha) = (x, B_{i,j}(x, \alpha))$, and $B^{-1}(x, \beta) = (x, A'(x, \beta))$ for $(x, \beta) \in O$. Then A' is a C^ω map from O to $\mathbf{R}^{n^2}$,

$$f(x + \langle \alpha, \partial f \rangle) = f(x) + \sum_{i,j}^{n} B_{i,j}(x, \alpha) \frac{\partial f}{\partial x_i}(x) \frac{\partial f}{\partial x_j}(x) \quad \text{for } (x, \alpha) \in W,$$

$$f(x + \langle A'(x, \beta), \partial f \rangle) = f(x) + \sum_{i,j}^{n} \beta_{i,j} \frac{\partial f}{\partial x_i}(x) \frac{\partial f}{\partial x_j}(x) \quad \text{for } (x, \beta) \in O.$$

Choose $\Phi = (\phi_{i,j})$ so small that its graph is contained in O. Then $\pi(x) = x + \langle A'(x, \Phi(x)), \partial f \rangle$ fulfills the requirements in (ii). Here if $\phi_{i,j}$ are small in the Whitney C^r or the strong Whitney C^∞ topology, π is close to id in the respective topology.

If f, M and v_i are of class C^ω and if ϕ is of class C^∞, the same arguments as above work and the diffeomorphism of equivalence is of class C^∞. Thus we complete the proof of (ii) in the analytic case. Point (ii) in the C^∞ or Nash case follows also from the same proof. The difference is only that the existence of C^∞ or Nash $g_{i,j}$ follows from a partition of unity of class C^∞ or theorem 2.8, respectively.

Consider (iii). Assume that M is not compact. Let M be embedded in a Euclidean space so that its closure is a compact Nash manifold with boundary. Now, we consider an open semialgebraic tubular neighborhood of $\overline{M}$ and extend f to the neighborhood as before. Then we can assume that M is open in $\mathbf{R}^n$ and $\overline{M}$ is a compact Nash manifold with corners, and for the construction of π it suffices to see that ϕ is of the form $\sum_{i,j}^{n} \phi_{i,j} \frac{\partial f}{\partial x_i} \frac{\partial f}{\partial x_j}$ for some Nash functions $\phi_{i,j}$ on M r'-flat at X, where $0 \ll r' \ll r \in \mathbf{N}$. Actually, assume that there exist such $\phi_{i,j}$. Then by the above proof, we only need to find small semialgebraic $C^{r''}$ functions $\phi'_{i,j}$ on M in the semialgebraic $C^{r''}$ topology such that $\phi'_{i,j} = \phi_{i,j}$ on some semialgebraic neighborhood of X for $0 < r'' \ll r' \in \mathbf{N}$.

Consider only the case $r'' = 1$ because the general case can be proved in the same way. Set $g(x) = \prod_{a \in f(X)} (f(x) - a)^2$, and let h be a Nash function on M extensible to a Nash function $\overline{h}$ on $\overline{M}$ such that $0 < h \le 1/2$, such that (1) $|\frac{\partial h}{\partial x_k}| \le 1$, for $k = 1, ..., n$, and $\overline{h}^{-1}(0) = \overline{M} - M$, which exists since $\overline{M}$ is a compact Nash manifold with corners. Let $\psi(t)$ be a semialgebraic C^1 function on $\mathbf{R}$ such that $0 \le \psi \le 1$

and $\psi = 1$ on $(-\infty, 1]$ whereas $\psi = 0$ on $[2, \infty)$. Then $\phi'_{i,j} = \phi_{i,j}\psi(g/h^m)$ fulfill the requirements for some $m \in \mathbf{N}$. Actually, Clearly $\phi'_{i,j} = \phi_{i,j}$ on a semialgebraic neighborhood $\{x \in M : g(x) \le h^m(x)\}$ of X in M, and $\phi'_{i,j} = 0$ on $\{g(x) \ge 2h^m(x)\}$. Hence we prove that each $\phi'_{i,j}$ is small on $V \overset{\text{def}}{=} \{g(x) \le 2h^m(x)\}$ in the semialgebraic C^1 topology. Let $\epsilon > 0 \in \mathbf{R}$. Let ξ denote the Nash function on M defined to be $\phi_{i,j}/g^2$ on $M - X$ and 0 on X. Then ξ, $\frac{\partial g}{\partial x_k}$ and $\frac{\partial \phi_{i,j}}{\partial x_k}$, $k = 1, ..., n$, vanish at X. Hence there exists a semialgebraic neighborhood W of X in M where

$$(2)\ |\phi_{i,j}| \le \epsilon g^2, \quad (3)\ |\frac{\partial g}{\partial x_k}| \le 1, \quad (4)\ |\frac{\partial \phi_{i,j}}{\partial x_k}| \le \epsilon.$$

By the Łojasiewicz inequality, we have $V \subset W$ for large m. Note that (5) $g \le 1/2^{m-1}$ on V since $h \le 1/2$. Set $c = \max|\frac{d\psi}{dt}|$. Then on V

$$|\phi'_{i,j}| = |\phi_{i,j}\psi(\frac{g}{h^m})| \overset{(2)}{\le} \epsilon g^2 \overset{(5)}{<} \epsilon,$$

$$|\frac{\partial \phi'_{i,j}}{\partial x_k}| \le |\frac{\partial \phi_{i,j}}{\partial x_k}\psi(\frac{g}{h^m})| + |\phi_{i,j}\frac{d\psi}{dt}(\frac{g}{h^m})|(|\frac{\partial g}{\partial x_k}|/h^m + m|g\frac{\partial h}{\partial x_k}|/h^{m+1}),$$

$$|\frac{\partial \phi_{i,j}}{\partial x_k}\psi(\frac{g}{h^m})| \overset{(4)}{\le} \epsilon,$$

$$|\phi_{i,j}\frac{d\psi}{dt}(\frac{g}{h^m})\frac{\partial g}{\partial x_k}|/h^m \overset{(3)}{\le} \frac{c|\phi_{i,j}|}{h^m} \overset{\text{by def. of } V}{\le} \frac{2c|\phi_{i,j}|}{g} \overset{(2),(5)}{\le} c\epsilon,$$

$$m|\phi_{i,j}\frac{d\psi}{dt}(\frac{g}{h^m})g\frac{\partial h}{\partial x_k}|/h^{m+1} \overset{(1)}{\le} \frac{mc|\phi_{i,j}|g}{h^{m+1}} \overset{(2)}{\le} 2^{\frac{m+1}{m}} mc\epsilon g^{2-\frac{1}{m}} \overset{(5)}{\le} 2^{4+\frac{1}{m}-2m} mc\epsilon.$$

Hence $\phi'_{i,j}$ is small on V for large m.

It remains to find $\phi_{i,j}$. Let $\mathcal{K}$ denote the sheaf of $\mathcal{N}$-ideals on M defined by X. Then ϕ is a cross-section of $\mathcal{K}^r$ since ϕ is r-flat at X and since X is normal crossing. On the other hand, $\sum_{i=1}^n \frac{\partial f}{\partial x_i}\mathcal{N} \supset \mathcal{K}^{r'}$ since f has only normal crossing singularities. Hence ϕ is a cross-section of $\sum_{i,j=1}^n \frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j}\mathcal{K}^{r'}$ because of $r' \ll r$. Let g_l, for $l = 1, ..., k'$, be global generators of $\mathcal{K}^{r'}$ (theorem 2.7). Apply theorem 2.8 to the surjective $\mathcal{N}$-homomorphism that assigns to $(\alpha_{i,j,l}) \in \mathcal{N}_a^{n^2k'} \subset \mathcal{N}^{n^2k'}$, for $a \in M$, the value

$$\sum \alpha_{i,j,l}g_{la}(\frac{\partial f}{\partial x_i})_a(\frac{\partial f}{\partial x_j})_a \in \sum_{i,j=1}^n (\frac{\partial f}{\partial x_i})_a(\frac{\partial f}{\partial x_j})_a\mathcal{K}_a^{r'} \subset \sum_{i,j=1}^n \frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j}\mathcal{K}^{r'}.$$

Then there exist Nash functions $\phi_{i,j}$, for $i,j = 1, ..., n$, in $H^0(M, \mathcal{K}^{r'})$ such that $\phi = \sum_{i,j=1}^n \phi_{i,j}\frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j}$. It follows that $\phi_{i,j}$ are r'-flat at X.

The case of compact M is clear by the above arguments. $\square$

Proposition 4.9. *(Compactification of a Nash function with only normal crossing singularities) Let f be a bounded Nash function on a non-compact Nash manifold M with only normal crossing singularities. Then there exist a compact Nash manifold with corners M' and a Nash diffeomorphism $\pi : M \to \operatorname{Int} M'$ such that $f \circ \pi^{-1}$ is extensible to a Nash function on M' with only normal crossing singularities.*

The analytic case does not necessarily hold.

We cannot necessarily choose M' with smooth boundary. For example any compact Nash manifold with boundary whose interior is Nash diffeomorphic to $M = \mathbf{R}^3$ is Nash diffeomorphic to a closed ball in $\mathbf{R}^3$ (Theorem VI.2.2 in [S_2]). But there does not exist Nash function on a 2-sphere with only normal crossing singularities (see remark (v) after theorem 3.2).

Extensibility of a Nash function to a compact Nash manifold with corners is shown in Proposition VI.2.8 in [S_2]. Hence the problem is to impose to the extension to have only normal crossing singularities.

Proof of proposition 4.9. Set $n = \dim M$, and $X = f^{-1}(f(\operatorname{Sing} f))$, set $B^N = \{x \in \mathbf{R}^N : |x| \le 1\}$ for a positive integer N and $S^{N-1} = \partial B^N$. Since there exists a Nash embedding of M into $\mathbf{R}^N$ such that the image is closed in $\mathbf{R}^N$, we can assume by lemma 4.6 that $M \subset \operatorname{Int} B^N$, that $\overline{M} - M \subset S^{N-1}$, that $\overline{M}$ is a compact Nash manifold with boundary, and moreover $\overline{M}$ intersects transversally with S^{N-1} in the sense that some Nash manifold extension $\tilde{M}$ of $\overline{M}$ intersects transversally with S^{N-1}, that $\overline{X}$ is a normal crossing Nash subset of $\overline{M}$, and there exists a Nash function g on $\tilde{M}$ with only normal crossing singularities such that $g(\operatorname{Sing} g) = f(\operatorname{Sing} f)$, the equality $g^{-1}(g(\operatorname{Sing} g)) \cap \overline{M} = \overline{X}$ holds, and such that $g = f$ on X, for each $a \in X$, $g(x) - g(a)$ has the same multiplicity as $f(x) - f(a)$ at a and $g(b) > g(a)$ if and only if $f(b) > f(a)$ for $b \in M$. We do not know whether $g|_M$ is Nash right equivalent to f. We will modify $\overline{M}$ and g so that this is indeed the case and so that $g|_{\overline{M}}$ has only normal crossing singularities.

Let ϕ be a polynomial function on $\mathbf{R}$ such that $\phi^{-1}(0) = f(\operatorname{Sing} f)$ and ϕ is regular at $\phi^{-1}(0)$. Let $r \in \mathbf{N}$ be large enough. Apply lemma 4.7 to $\phi \circ f$ and $\phi \circ g|_M$. Then we have a Nash diffeomorphism τ_1 of M such that $\tau_1(X) = X$ and $f \circ \tau_1 - g|_M$ is r-flat at X. Hence replacing f with $f \circ \tau_1$, we assume that $f - g$ is r-flat at X. Next, by proposition 4.8,(iii) there exists a semialgebraic C^r diffeomorphism τ_2 of M such that $g = f \circ \tau_2$ on a semialgebraic neighborhood V of X in M and τ_2 is of class Nash on V. We can choose V of the form $\{x \in M : \phi^2 \circ g(x) \le c(x)\xi^m(x)\}$ by the Łojasiewicz inequality, where $\xi(x) = (1 - |x|^2)/2$ for $x \in \tilde{M}$, where c is a positive Nash function on $\tilde{M}$ such that c depends on only $|x|$ and m is a large odd integer. Shrink $\tilde{M}$ so that $\xi < 0$ on $\tilde{M} - \overline{M}$. We can choose, moreover, c and m so that $\phi^2 \circ g - c\xi^m$ is regular at $A - S^{N-1}$, where A denotes the zero set of $\phi^2 \circ g - c\xi^m$, and hence V is a Nash manifold with boundary $\{x \in M : \phi^2 \circ g(x) = c(x)\xi^m(x)\}$. Actually, let $0 < \epsilon_0 \in \mathbf{R}$ be small. Then for any $0 < \epsilon \in \mathbf{R}$ with $\epsilon < \epsilon_0$, $\xi^{-1}(\epsilon) \cup (\phi \circ g)^{-1}(0)$ is normal crossing in $\tilde{M}$, and hence for small c and large m, the function $\phi^2 \circ g$ on $\{x \in \xi^{-1}(\epsilon) : 0 < \phi^2 \circ g(x) < 2c(x)\xi^m\}$ is regular. We can choose c and m independently of ϵ. Therefore, $\phi^2 \circ g - c\xi^m$ is regular at $A \cap \xi^{-1}((0, \epsilon_0))$. Moreover, if we choose c and m so that $c\xi^m$ is close to a small constant on $M - \xi^{-1}((0, \epsilon_0/2])$, then $\phi^2 \circ g - c\xi^m$ is regular at $A - \xi^{-1}((0, \epsilon_0/2])$. Hence $\phi^2 \circ g - c\xi^m$ can be regular at $A - S^{N-1}$. However, we omit c for simplicity of notation. We want first to modify M so that $\overline{V}$ is a neighborhood of $\overline{X}$ in $\overline{M}$.

Apply theorem 2.10 to the two sheaves of $\mathcal{N}$-ideals on $\tilde{M}$ defined by $(\phi \circ g)^{-1}(0)$ and generated by $\xi \cdot (\phi^2 \circ g - \xi^m)$. Note that the former sheaf is normal crossing, the stalk of the latter is not generated by one regular function germ at a point of $\overline{X} - X$ only, and at least one of the two stalks of both sheaves at each $x \notin \overline{X} - X$ is $\mathcal{N}_x$. Then we have a composition of a finite sequence of blowings-up $\tau_3 : \hat{M} \to \tilde{M}$ along smooth Nash centers such that $\tau_3|$ [illegible] $: \tau_3^{-1}(\tilde{M} - (\overline{X} - X)) \to$

$\tilde{M} - (\overline{X} - X)$ is a Nash diffeomorphism and $(\phi \circ g \cdot \xi \cdot (\phi^2 \circ g - \xi^m)) \circ \tau_3$ has only normal crossing singularities at its zero set, say Y. It follows that $(\hat{M}, Y, \overline{\tau_3^{-1}(M)})$ is Nash diffeomorphic to $(\mathbf{R}^n, \{(x_1, ..., x_n) \in \mathbf{R}^n : x_1 \cdots x_{n'} = 0\}, B)$ locally at each point of $\overline{\tau_3^{-1}(M)}$ for some $n' \, (\le n) \in \mathbf{N}$, where B denotes the closure of the union of some connected components of $\{x_1 \cdots x_{n'} \neq 0\}$, and $\overline{\tau_3^{-1}(M)} - \tau_3^{-1}(\overline{X} - X)$ is a Nash manifold with boundary. However, $\overline{\tau_3^{-1}(M)}$ is not necessarily a manifold with corners. It may happens that $\tau_3^{-1}(M)$ is locally diffeomorphic to the union of more than one connected components of $\{(x_1, ..., x_n) \in \mathbf{R}^n : x_1 \cdots x_{n'} \neq 0\}$ at some point of $\tau_3^{-1}(\overline{X} - X)$, for $0 < n' \, (\le n) \in \mathbf{N}$. Then we need to separate these connected components. That is possible as shown in the proof of Theorem VI.2.1 in [S$_2$]. Namely, there exist a compact Nash manifold L with corners and a Nash immersion $\tau_4 : L \to \overline{\tau_3^{-1}(M)}$ such that $\tau_4|_{L - \mathrm{Sing}\, \partial L}$ is a Nash diffeomorphism to its image and the image contains $\overline{\tau_3^{-1}(M)} - \tau_3^{-1}(\overline{X} - X) \, (\supset \tau_3^{-1}(M))$.

Clearly $(\phi \circ g \cdot \xi \cdot (\phi^2 \circ g - \xi^m)) \circ \tau_3 \circ \tau_4$ has only normal crossing singularities at its zero set $\tau_4^{-1}(Y)$ since τ_4 is an immersion. Set $\tau = \tau_2 \circ \tau_3 \circ \tau_4|_{\mathrm{Int}\, L}$ and $h = g \circ \tau_3 \circ \tau_4$. Define $W = (\tau_3 \circ \tau_4)^{-1}(V)$ and $W' = \overline{W} - \overline{\partial W}$ and set $Z = (\tau_3 \circ \tau_4)^{-1}(X)$. Then W is a non-compact Nash manifold with boundary; τ is a semialgebraic C^r diffeomorphism from $\mathrm{Int}\, L$ to M and of class Nash on W; h is a Nash function on L; $h = f \circ \tau$ on W; h is regular on $\mathrm{Int}\, L - Z$; $h|_{\mathrm{Int}\, L \cup W'}$ has only normal crossing singularities at $\overline{Z}$ though h is not necessarily so globally; $\overline{W}$ is a neighborhood of $\overline{Z}$ in L because if it were not, $\overline{Z} \cap \overline{(\tau_3 \circ \tau_4)^{-1}(\{x \in M : \phi^2 \circ g(x) = \xi^m(x)\})}$ could be not empty and of dimension $n - 2$ but contained in $(\tau_3 \circ \tau_4)^{-1}(\xi^{-1}(0))$, which contradicts the normal crossing property of $(\phi \circ g \cdot \xi \cdot (\phi^2 \circ g - \xi^m)) \circ \tau_3 \circ \tau_4$. Note that W' and $\overline{W}$ are Nash manifolds with corners by the next fact and the normal crossing property of $(\xi \cdot (\phi^2 \circ g - \xi^m)) \circ \tau_3 \circ \tau_4$. Thus $\overline{V}$ is changed to $\overline{W}$—a neighborhood of $\overline{Z}$ in L. We consider h on L in place of g on $\overline{M}$.

We replace τ by a Nash diffeomorphism. Let $0 \ll r \in \mathbf{N}$, set $\overline{\psi} = (\phi^r \circ h \cdot \xi^r) \circ \tau_3 \circ \tau_4$ on L and $\psi = \overline{\psi}|_{\mathrm{Int}\, L}$, and let $\mathcal{I}$ denote the sheaf of $\mathcal{N}$-ideals on $\mathrm{Int}\, L$ generated by ψ. Then we regard τ as an element of $H^0(\mathrm{Int}\, L, \mathcal{N}/\mathcal{I})^N$ because $\mathrm{supp}\, \mathcal{N}/\mathcal{I} = Z$ and τ is of class Nash near there. Hence by theorem 2.8 there exists a Nash map $\tau' : \mathrm{Int}\, L \to \mathbf{R}^N$ such that $\tau - \tau' = \psi\theta$ for some semialgebraic C^r map $\theta : \mathrm{Int}\, L \to \mathbf{R}^N$ of class Nash on W. Approximate θ by a Nash map $\theta' : \mathrm{Int}\, L \to \mathbf{R}^N$ in the semialgebraic C^r topology, and set $\tau'' = p \circ (\tau' + \psi\theta')$, where p denotes the orthogonal projection of a semialgebraic tubular neighborhood of M in $\mathbf{R}^N$. Then τ'' is a well-defined Nash diffeomorphism from $\mathrm{Int}\, L$ to M and close to τ in the semialgebraic C^r topology; $f \circ \tau'' - h|_{\mathrm{Int}\, L} = \psi\delta$ for some semialgebraic C^r function δ on $\mathrm{Int}\, L$ though $f \circ \tau'' - h|_{\mathrm{Int}\, L}$ does not necessarily vanish on W; moreover, δ is extensible to a semialgebraic $C^{r'}$ function $\overline{\delta}$ on $\mathrm{Int}\, L \cup W'$ for $0 \ll r' \, (\ll r) \in \mathbf{N}$ by the definition of the semialgebraic $C^{r'}$ topology, by the fact that a small semialgebraic C^r function on $\mathrm{Int}\, L$ is extensible to a semialgebraic C^r function on L and by

$$f \circ \tau'' - h|_{\mathrm{Int}\, L} = f \circ p \circ (\tau + \psi \cdot (\theta' - \theta)) - f \circ p \circ \tau \quad \text{ on } W.$$

The last equality implies also that δ is of class Nash on W, and hence on $\mathrm{Int}\, L$ since $f \circ \tau''$ and h are Nash functions and $\psi^{-1}(0) \subset W$.

Next we modify h. Let $\overline{\delta}'$ be a Nash approximation on $\mathrm{Int}\, L \cup W'$ of $\overline{\delta}$ in the semialgebraic $C^{r'}$ topology and set $\delta' = \overline{\delta}'|_{\mathrm{Int}\, L}$ and $h' = h + \psi\overline{\delta}'$ on $\mathrm{Int}\, L \cup W'$. Then

h' is a Nash function on $\mathrm{Int}\, L\cup W'$ and has only normal crossing singularities at $\overline{Z}$ by the same property of $h|_{\mathrm{Int}\, L\cup W'}$ and by the definition of h', and $f\circ\tau''-h'|_{\mathrm{Int}\, L}$ is of the form $\psi\cdot(\delta-\delta')$. Hence f and $h'\circ\tau''^{-1}$ satisfy the conditions in proposition 4.8,(ii) because $\phi^r\circ h\circ\tau''^{-1}$ is of the form $\sum_{i,j=1}^k\psi_{i,j}v_f\cdot v_jf$ for some Nash functions $\psi_{i,j}$ on M and Nash vector fields v_i, for $i=1,...,k$, on M which span the tangent space of M at each point of M and because $\xi^r|_M\cdot(\delta-\delta')\circ\tau''^{-1}$ is small as a semialgebraic $C^{r'}$ function on M. Consequently, f and $h'\circ\tau''^{-1}$ are Nash right equivalent, and we can replace f with $h'|_{\mathrm{Int}\, L}$.

We can assume that $W'\cap\partial L$ is the union of some connected components σ of strata of the canonical stratification $\{L_i\}$ of ∂L such that $\overline{\sigma}\cap\overline{Z}\neq\emptyset$. Actually, let ψ_L be a non-negative Nash function on L with zero set $\overline{Z}$, and let $\epsilon>0\in\mathbf{R}$ be such that the restriction of ψ_L to $\psi_L^{-1}((0,2\epsilon))$ is regular. Then $\psi_L^{-1}(\epsilon)$ is a compact Nash manifold with corners equal to $\partial L\cap\psi_L^{-1}(\epsilon)$. Let $\{L_{\epsilon,i}\}$ denote the canonical stratification of $\partial L\cap\psi_L^{-1}(\epsilon)$. We blow up $L_{\epsilon,i}$ as follows. Let L' and $\tilde L'$ be a compact Nash submanifold possibly with corners of L and some Nash manifold extension of L' respectively. If $\tilde L'\cap L=L'$ and $(L,\tilde L')$ is locally diffeomorphic to $(\{(x_1,...,x_n)\in\mathbf{R}^n: x_1\ge 0,...,x_{n'}\ge 0\},\{x_{n_1}=\cdots=x_{n_k}=0\})$ for some $n'\,(\le n), 1\le n_1<\cdots<n_k\le n\in\mathbf{N}$, then we say L' has the property $(*)$. For L' with $(*)$, consider $\gamma:\Gamma\to L$—the restriction of the blowing-up of a small Nash manifold extension $\tilde L$ of L along center $\tilde L\cap\tilde L'$ to the closure of inverse image of $L-L'$, modify $\gamma:\Gamma\to L$ so that Γ is a compact Nash manifold with corners by the idea in the proof of Theorem VI.2.1 in [S$_2$] as before, use the same notation $\gamma:\Gamma\to L$, and call it the $(*)$-blowing-up of L along center L'. Note that $\gamma^{-1}(L')$ is the closure of the union of some connected components of $\mathrm{Reg}\,\partial\Gamma$. Set $\Gamma_{-1}=L$ and let $0\le k\le n-2$. Inductively we define $(*)$-blowing-up $\gamma_k:\Gamma_k\to\Gamma_{k-1}$ of Γ_{k-1} along center $L_{\epsilon,0}$ if $k=0$ and along center $\overline{(\gamma_0\circ\cdots\circ\gamma_{k-1})^{-1}(L_{\epsilon,k})}$ if $k>0$, which is possible because $L_{\epsilon,0}$ and $\overline{(\gamma_0\circ\cdots\circ\gamma_k)^{-1}(L_{\epsilon,k+1})}$ for $0\le k\le n-3$ are compact Nash submanifolds with corners of Γ_{-1} and Γ_k with $(*)$, respectively. Thus we assume that the above condition on W holds considering $(\Gamma_{n-2},(\psi_L\circ\gamma_0\circ\cdots\gamma_{n-2})^{-1}([0,\epsilon])-\partial\Gamma_{n-2})$ in place of (L,W). Here we choose ϵ so small that $(**)$ h' is extensible to a Nash function on an open semialgebraic neighborhood of $\mathrm{Int}\, L\cup\overline{W}$ in L with only normal crossing singularities.

Moreover, we can assume that the closure of each connected component of $\mathrm{Reg}\,\partial L$ is a Nash manifold possibly with corners. Indeed, we obtain this situation if we repeat the same arguments as above to the canonical stratification of ∂L compatible with $\{x\in\partial L:\mathrm{dis}(x,L_k)=\epsilon_k,\ \mathrm{dis}(x,L_i)\ge\epsilon_i,\ i=0,...,k-1\}$, for $k=0,...,n-2$. Here we naturally define the canonical stratification of ∂L compatible with the above family in the same way as in the remark after the proof of lemma 4.5. After this modification of L, the property $(**)$ continues to hold.

Let M_j, for $j\in J$, be the set of closures of the connected components of $\mathrm{Reg}\,\partial L$, and let J_0 denote the subset of J consisting of j such that $M_j\cap\overline{Z}=\emptyset$. Let $\tilde L$ and $\tilde M_j$ be Nash manifold extensions of L and M_j, respectively, which are contained and closed in a small open semialgebraic neighborhood U of L in the ambient Euclidean space such that $\cup_{j\in J}\tilde M_j$ is normal crossing in $\tilde L$ and for each $j\in J$ there is one and only one connected component of $\tilde L-\tilde M_j$ which does not intersect with L. Let $\tilde Z$ denote the smallest Nash subset of $\tilde L$ containing Z. Then $\tilde Z$ is normal crossing in $\tilde L$, and there exist Nash functions ϕ_j on $\tilde L$ with zero set $\tilde M_j$, regular there and

with $\chi_j > 0$ on $\operatorname{Int} L$.

By $(**)$ we can choose a sufficiently small U so that h' can be extended to a Nash function h'_+ on $L_+ \overset{\text{def}}{=} \{x \in \tilde{L} : \chi_j(x) > 0,\ j \in J_0\}$, such that $h'_+(\operatorname{Sing} h'_+) = h'(Z)$ and h'_+ has only normal crossing singularities. Now we smooth h'_+ at $\overline{L_+} - L_+$ as in the proof of Proposition VI.2.8 in [S_2]. Let $\tilde{L} \subset \mathbf{R}^N$, set $G = \operatorname{graph} h'_+ \subset L_+ \times \mathbf{R}$, and let G^Z be the Zariski closure of G in $\mathbf{R}^N \times \mathbf{R}$ and Q be the normalization of G^Z in $\mathbf{R}^N \times \mathbf{R} \times \mathbf{R}^{N'}$ for some $N' \in \mathbf{N}$, and let $r : Q \to \mathbf{R}^N \times \mathbf{R}$ and $q : Q \to \mathbf{R}^N$ denote the restrictions to Q of the projections $\mathbf{R}^N \times \mathbf{R} \times \mathbf{R}^{N'} \to \mathbf{R}^N \times \mathbf{R}$ and $\mathbf{R}^N \times \mathbf{R} \times \mathbf{R}^{N'} \to \mathbf{R}^N$, respectively. Then it is known that r is a proper map to G^Z, and by Artin-Mazur Theorem there exists a union of connected components R of $Q - r^{-1}(\overline{G} - G)$ such that $R \subset \operatorname{Reg} Q$ and $r|_R$ is a Nash diffeomorphism onto G. Here we can replace $r^{-1}(\overline{G} - G)$ with a Nash subset $q^{-1}((\prod_{j \in J_0} \chi_j)^{-1}(0))$ of Q because $r^{-1}(\overline{G} - G) \subset q^{-1}((\prod_{j \in J_0} \chi_j)^{-1}(0))$ and $R \cap q^{-1}((\prod_{j \in J_0} \chi_j)^{-1}(0)) = \emptyset$; $q|_R$ is a Nash diffeomorphism onto L_+; the map $h'_+ \circ q|_R$ is the restriction of the projection $\mathbf{R}^N \times \mathbf{R} \times \mathbf{R}^{N'} \to \mathbf{R}$ and hence extensible to a smooth rational function on Q; the set $\overline{R \cap q^{-1}(L)}$ is compact because r is proper and because $\overline{G \cap L \times \mathbf{R}}$ is compact by boundedness of f; the function $h'_+ \circ q|_R$ has only normal crossing singularities because the same is true for h'_+. However, $\chi_j \circ q$ are now not necessarily regular at their zero sets. By theorem 2.8, $R \cap q^{-1}(\tilde{Z})$ is a Nash subset of $\operatorname{Reg} Q$ and there exists a Nash function α on $\operatorname{Reg} Q$ whose zero set is $R \cap q^{-1}(\tilde{Z})$ and which has only normal crossing singularities there since $R \cap q^{-1}(\tilde{Z})$ is a Nash subset of R and since its closure in $\operatorname{Reg} Q$ does not intersect with $\overline{R} - R$.

Thus replacing $\tilde{L},\ L_+,\ h'_+,\ \chi_j$ and $\tilde{Z}$ with $\operatorname{Reg} Q,\ R,\ h'_+ \circ q|_R,\ \chi_j \circ q|_{\operatorname{Reg} Q}$ and $R \cap q^{-1}(\tilde{Z})$ we assume from the beginning that M and f satisfy moreover the following conditions.

(i) $\tilde{f}$ and χ_j, for $j \in J$, are a finite number of Nash functions on a Nash manifold $\tilde{M}$, and $\tilde{X}$ is a normal crossing Nash subset of $\tilde{M}$.
(ii) M is the union of some connected components of $\tilde{M} - (\prod_{j \in J} \chi_j)^{-1}(0)$, the set $\overline{M}$ is compact, the equalities $f = \tilde{f}|_M$ and $X = \tilde{X} \cap M$ hold (we do not assume that $\overline{M}$ is a manifold with corners).

We make $\prod_{j \in J} \chi_j$ normal crossing at its zero set. Apply theorem 2.10 to the sheaf of $\mathcal{N}$-ideals on $\tilde{M}$ defined by $\tilde{X}$ and the sheaf of $\mathcal{N}$-ideals $\prod_{j \in J} \chi_j \mathcal{N}$. Then via blowings-up, $\prod_{j \in J} \chi_j$ becomes to have only normal crossing singularities at its zero set, and the conditions (i) and (ii) do not change because the subset of $\tilde{M}$ where we modify by blowings-up is contained in $(\prod_{j \in J} \chi_j)^{-1}(0)$.

It remains to make $\tilde{f}$ together with $(\prod_{j \in J} \chi_j)^{-1}(0)$ normal crossing. Let $\{\tilde{M}_i\}$ denote the canonical stratification of $(\prod_{j \in J} \chi_j)^{-1}(0)$, set $\tilde{M}_n = \tilde{M} - (\prod_{j \in J} \chi_j)^{-1}(0)$, and let $\tilde{\phi}$ be a polynomial function on $\mathbf{R}$ such that $\tilde{\phi}^{-1}(0) = \cup_{i=0}^n \tilde{f}(\operatorname{Sing} \tilde{f}|_{\tilde{M}_i})$ and $\tilde{\phi}$ is regular at $\tilde{\phi}^{-1}(0)$. Once more, apply theorem 2.10 to the sheaf of $\mathcal{N}$-ideals on $\tilde{M}$ defined by $\tilde{X} \cup (\prod_{j \in J} \chi_j)^{-1}(0)$ and the sheaf of $\mathcal{N}$-ideals $[\tilde{\phi} \circ \tilde{f} \mathcal{N} : \cap_i \mathcal{I}_i^{\alpha_i}] \overset{\text{def}}{=} \cup_{x \in \tilde{M}} \{\rho \in \mathcal{N}_x : \rho \cap_i \mathcal{I}_{ix}^{\alpha_i} \subset \tilde{\phi} \circ \tilde{f} \mathcal{N}_x\}$, where $\cap_i \mathcal{I}_i$ is the decomposition of the sheaf of $\mathcal{N}$-ideals on $\tilde{M}$ defined by $\tilde{X}$ to irreducible finite sheaves of $\mathcal{N}$-ideals and each α_i is the maximal integer such that $\tilde{\phi} \circ \tilde{f} \mathcal{N}$ is divisible by $\mathcal{I}_i^{\alpha_i}$. Then $(\tilde{f} - \tilde{f}(x_0)) \prod_{j \in J} \chi_j$ becomes to have only normal crossing singularities at its zero set for each $x_0 \in \tilde{M}$

and the subset of $\tilde M$ where we modify now by blowings-up does not intersect with M because the stalk of the latter sheaf at each point of M is generated by a regular function germ and because

$$(\tilde X \cup (\prod_{j\in J} \chi_j)^{-1}(0)) \cap \operatorname{supp} \mathcal{N}/[\tilde\phi \circ \tilde f \mathcal{N} : \cap_i \mathcal{I}_i^{\alpha_i}] \cap M = \emptyset.$$

Finally, we separate as before M at the points of $\overline{M}$ where M is not locally connected so that $\overline{M}$ is a compact Nash manifold with corners. Then $\tilde f|_{\overline{M}}$ has only normal crossing singularities, and we complete the proof. □

5. Proofs of theorem 3.2 and theorems 3.1,(2) and 3.1,(3)

5.1. Proof of theorem 3.2.

By proposition 4.1 it suffices to prove the Nash case and, moreover, that the cardinality of Nash R-L equivalence classes of Nash functions with only normal crossing singularities on a compact Nash manifold possibly with corners is zero or countable. The reasons are that first we can restrict functions to being bounded by the fact that $\mathbf{R}$ is Nash diffeomorphic to $(0, 1)$ and secondly by proposition 4.9 we can regard a non-compact Nash manifold M and a bounded Nash function f with only normal crossing singularities on M as the interior of a compact Nash manifold with corners M' and the restriction to M of a Nash function on M' with only normal crossing singularities. Assume that there is at least one Nash function f on M with only normal crossing singularities. Then the cardinality is infinite because we can increase arbitrarily the cardinality of the critical value set, which is finite, by replacing f with $\pi \circ f$ for some Nash function π on $\mathbf{R}$. Let $\{X_\alpha\}_{\alpha\in A}$ denote all normal crossing Nash subsets of M. We define α and α' in A to be equivalent if there exists a Nash diffeomorphism of M which carries X_α to $X_{\alpha'}$. Then by lemma 4.4 the cardinality of equivalence classes of A is countable. Hence it suffices to see that for each X_α there exist at most a countable number of Nash R-L equivalence classes of Nash functions f on M with only normal crossing singularities such that $f^{-1}(f(\operatorname{Sing} f)) = X_\alpha$. Let F_α denote all such Nash functions. Clearly there are a finite number of equivalence classes of $\{f|_{X_\alpha} : X_\alpha \to \mathbf{R} : f \in F_\alpha\}$ under the Nash left equivalence relation since the value sets are finite. Moreover, there are at most a countable number of choices of multiplicity of $f - f(a)$ at a for $f \in F_\alpha$ and $a \in X_\alpha$. Hence we reduce the problem to the following one. Fix $f \in F_\alpha$, and let F_f denote the family of $g \in F_\alpha$ such that $g = f$ on X_α and $g - g(a)$ has the same multiplicity as $f - f(a)$ at each point a of X_α. Then the cardinality of Nash right equivalence classes of functions in F_f is finite. Moreover, it suffices to prove that each element of F_f, say f, is stable in F_f in the sense that any $g \in F_f$ near f in the C^∞ topology is Nash right equivalent to f because there are only a finite number of connected components in F_f.

Set $n = \dim M$, embed M in $\mathbf{R}^N$, and let $\{M_i\}$ denote the canonical stratification of M. There exist Nash vector fields $v_1, ..., v_k$ on M such that $v_{1x}, ..., v_{kx}$ span the tangent space $T_x M_i$ of M_i at each $x \in M_i$. If we regard M as $\{(x_1, ..., x_n) \in \mathbf{R}^n : x_1 \geq 0, ..., x_{n'} \geq 0\}$ by its local coordinate system, then $x_i \frac{\partial}{\partial x_i}$ is contained in the linear space over $N(M)$ spanned by $v_1, ..., v_k$ for each $1 \leq i \leq n'$. Actually, set $L_i = \cup_{j=0}^i M_j$, for $i = 0, ..., n-1$, and choose a Nash manifold extension $\tilde M$ of M and Nash subset extensions $\tilde L_i$ of L_i in $\tilde M$ so that $\tilde L_{n-1}$ is normal crossing

in $\tilde M$ and $\{\tilde L_i - \tilde L_{i-1}\}$ is the canonical stratification of $\tilde L_{n-1}$. Set $L_n = M$ and $\tilde L_n = \tilde M$ also. Then when we describe $(\tilde M, \tilde L_{n-1})$ by a local coordinate system as $(*)$ $(\mathbf{R}^n, \{(x_1, ..., x_n) \in \mathbf{R}^n : x_1 \cdots x_{n'} = 0\})$,

$$\tilde L_i = \bigcup_{1\le j_1<\cdots<j_{n-i}\le n'} \{(x_1, ..., x_n) \in \mathbf{R}^n : x_{j_1} = \cdots = x_{j_{n-i}} = 0\},\ n - n' \le i \le n.$$

We consider the situation on $\tilde M$ rather than on M because the existence of $v_1, ..., v_k$ follows from the existence of Nash vector fields on $\tilde M$ with the same properties.

First, let $w_{n,1}, ..., w_{n,k_n}$ be Nash vector fields on $\tilde M$ which span the tangent space of $\tilde M$ at each point, and α_n a global generator of the sheaf of $\mathcal{N}$-ideals on $\tilde M$ defined by $\tilde L_{n-1}$—we can choose $\tilde M$ so that α_n exists because M is a manifold with corners. Then $v_{n,1} = \alpha_n w_{n,1}, ..., v_{n,k_n} = \alpha_n w_{n,k_n}$ are Nash vector fields on $\tilde M$, span the tangent space of $\tilde M$ at each point of $\tilde M - \tilde L_{n-1}$ and vanish at $\tilde L_{n-1}$, and in the case $(*)$, for each $1 \le i \le n'$, $x_i \frac{\partial}{\partial x_i}$ on $\{(x_1, ..., x_n) \in \mathbf{R}^n : x_j \ne 0$ for $1 \le j \le n'$ with $j \ne i\}$ is contained in the linear space over the Nash function ring on the set spanned by $v_{n,1}, ..., v_{n,k_n}$.

Next fix $i < n$ and consider on $\tilde L_i$. Then it suffices to prove the following two statements.

(i) There exist Nash vector fields $v_{i,1}, ..., v_{i,k_i}$ on $\tilde L_i$—Nash cross-sections of the restrictions to $\tilde L_i$ of the tangent bundle of $\mathbf{R}^N$, i.e. the restrictions to $\tilde L_i$ of Nash vector fields on $\mathbf{R}^N$ by theorem 2.8—which span the tangent space of $\tilde L_i - \tilde L_{i-1}$ at its each point and vanish at $\tilde L_{i-1}$ and such that in the case of $(*)$ the condition on each irreducible component $\{(x_1, ..., x_n) \in \mathbf{R}^n : x_{j_1} = \cdots = x_{j_{n-i}} = 0\}$, same as on $\tilde M$, is satisfied for $1 \le j_1 < \cdots < j_{n-i} \le n'$; to be precise, for any $1 \le j \le n'$ other than $j_1, ..., j_{n-i}$, then $x_j \frac{\partial}{\partial x_j}$ on

$$\{(x_1, ..., x_n) \in \mathbf{R}^n : x_{j_1} = \cdots = x_{j_{n-i}} = 0,\ x_l \ne 0 \text{ if } l \in \{1, ..., n'\} \setminus \{j_1, ..., j_{n-i}, j\}\}$$

is contained in the linear space over the Nash function ring on the set spanned by $v_{i,1}, ..., v_{i,k_i}$.

(ii) Any Nash vector field on $\tilde L_i$ tangent to $\tilde L_j - \tilde L_{j-1}$ at its each point for $j \le i$ is extensible to a Nash vector field on $\tilde L_{i+1}$ tangent to $\tilde L_{i+1} - \tilde L_i$ at each its point.

Proof of (i). By considering the Zariski closure of $\tilde L_i$ and its normalization and by Artin-Mazur Theorem, we have a Nash manifold P_i and a Nash immersion $\xi_i : P_i \to \tilde L_i$ such that $\xi_i|_{P_i - \xi_i^{-1}(\tilde L_{i-1})}$ is a Nash diffeomorphism onto $\tilde L_i - \tilde L_{i-1}$. Note that $\xi_i^{-1}(\tilde L_{i-1})$ is normal crossing in P_i. Apply the same arguments to $(P_i, \xi_i^{-1}(\tilde L_{i-1}))$ as on $(\tilde M, \tilde L_{n-1})$. Here the difference is only that we need a finite number of global generators $\alpha_{i,1}, \alpha_{i,2}, ...$ of the sheaf of $\mathcal{N}$-ideals on P_i defined by $\xi_i^{-1}(\tilde L_{i-1})$. Then there exist Nash vector fields $w_{i,1}, ..., w_{i,k_i}$ on P_i with the corresponding properties, and they induce semialgebraic C^0 vector fields $v_{i,1}, ..., v_{i,k_i}$ on $\tilde L_i$ through ξ_i because $w_{i,1}, ..., w_{i,k_i}$ vanish on $\xi_i^{-1}(\tilde L_{i-1})$. Such $v_{i,1}, ..., v_{i,k_i}$ are of class Nash by the normal crossing property of $\tilde L_{n-1}$ in $\tilde M$ and satisfy the conditions in (i).

Proof of (ii). Let v be a Nash vector field on $\tilde L_i$ in (ii), and $\xi_{i+1} : P_{i+1} \to \tilde L_{i+1}$ the same as above. Then since ξ_{i+1} is an immersion, v pulls back a Nash cross-section w of the restriction to $\xi_{i+1}^{-1}(\tilde L_i)$ of the tangent bundle of the Nash manifold P_{i+1} and by theorem 2.8 we obtain a Nash vector field on P_{i+1} whose restriction

to $\xi_{i+1}^{-1}(\tilde{L}_i)$ is w. This vector field induces a Nash vector field of $\tilde{L}_{i+1}$ through ξ_{i+1}, which is an extension of v, by the same reason as in the proof of (i).

Let $g \in F_f$ near f. It suffices to see that f and g are C^ω right equivalent. Actually, assume that there exists a C^ω diffeomorphism π of M such that $f = g \circ \pi$. Let $\tilde{M}$ and $\tilde{L}_{n-1}$ be the same as above and so small that f and g are extensible to Nash functions $\tilde{f}$ and $\tilde{g}$ on $\tilde{M}$, respectively. Extend π to a C^ω diffeomorphism $\tilde{\pi} : U_1 \to U_2$ between open neighborhoods of M in $\tilde{M}$ so that $\tilde{\pi}(U_1 \cap \tilde{L}_{n-1}) \subset \tilde{L}_{n-1}$ and $\tilde{f} = \tilde{g} \circ \pi$. As above, let α_n be a global generator of the sheaf of $\mathcal{N}$-ideals on $\tilde{M}$ defined by $\tilde{L}_{n-1}$. Then $\alpha_n \circ \tilde{\pi} = \beta\alpha_n$ on U_1 for some positive C^ω function β on U_1. Consider the following equations in variables $(x, y, z) \in \tilde{M}^2 \times \mathbf{R}$.

$$f(x) - g(y) = 0 \quad \text{and} \quad \alpha_n(y) - z\alpha_n(x) = 0$$

Here the second equation means that if $x \in \tilde{L}_{n-1}$ then $y \in \tilde{L}_{n-1}$. Then $y = \tilde{\pi}(x)$ and $z = \beta(x)$ are C^ω solutions. Hence by Nash Approximation Theorem II, there exist Nash germ M $y = \pi'(x)$ and $z = \beta'(x)$ solutions on M, which are approximations of the germs of $\tilde{\pi}$ and β on M. Thus $\pi'|_M$ is a Nash diffeomorphism of M and $f = g \circ \pi'$ on M.

Now we show the C^ω right equivalence of f and g. Set $G(x,t) = (1-t)f(x) + tg(x)$ for $(x,t) \in M \times [0,\, 1]$. Then $G(x, 0) = f(x)$ and $G(x, 1) = g(x)$. Hence by the same reason as in the proof of theorem 3.1,(1) it suffices to find a C^ω vector field v on $M \times [0,\, 1]$ of the form $\frac{\partial}{\partial t} + \sum_{i=1}^k a_i v_i$ for some C^ω functions a_i on $M \times [0,\, 1]$ such that $vG = 0$ on $M \times [0,\, 1]$, i.e.,

$$(**) \qquad f - g = \sum_{i=1}^{k} a_i(v_i f + tv_i(g - f)).$$

Moreover, as shown there, we only need to solve this equations locally at each point (x_0, t_0) of $M \times [0,\, 1]$ since M is compact.

If $x_0 \notin X_\alpha$, then $(v_i f)(x_0) \neq 0$ for some i and hence we have solutions of $(**)$ $a_j = 0$ for $j \neq i$ and $a_i = (f-g)/(v_i f + tv_i(g-f))$ around (x_0, t_0) because $g - f$ and hence $tv_i(g-f)$ are small in the C^∞ topology.

Let $x_0 \in X_\alpha$. Then we can assume that $M = \{x = (x_1, ..., x_n) \in \mathbf{R}^n : |x| \leq 1,\, x_1 \geq 0, ..., x_{n'} \geq 0\}$ for some $n' \,(\leq n) \in \mathbf{N}$, that $x_0 = 0$ and $f(x) = x^\beta$ for some $\beta = (\beta_1, ..., \beta_n) \in \mathbf{N}^n$ with $|\beta| > 0$, that $k = n$ and $v_1 = x_1\frac{\partial}{\partial x_1}, ..., v_{n'} = x_{n'}\frac{\partial}{\partial x_{n'}}, v_{n'+1} = \frac{\partial}{\partial x_{n'+1}}, ..., v_n = \frac{\partial}{\partial x_n}$ and that $f - g = bx^\beta$ for some small C^ω function b on M by lemma 2.12. Let i be such that $\beta_i \neq 0$. Then $v_i f = \beta_i x^\beta / x_i$ and $v_i(f-g) = b\beta_i x^\beta/x_i + \frac{\partial b}{\partial x_i}x^\beta$ if $i > n'$, and $v_i f = \beta_i x^\beta$ and $v_i(f - g) = b\beta_i x^\beta + x_i\frac{\partial b}{\partial x_i}x^\beta$ if $i \leq n'$. In any case $(**)$ is solved as before. Thus theorem 3.2 is proved.

5.2. Proof of theorems 3.1,(2) and 3.1,(3).

Let us consider the case where M is a manifold without corners.

Proof of (2). Set $X = f^{-1}(f(\operatorname{Sing} f))$ and $Y = g^{-1}(g(\operatorname{Sing} g))$, and let π be a C^2 diffeomorphism of M such that $f \circ \pi = g$. Then X and Y are normal crossing, $\pi(Y) = X$, and we assume that π is close to id in the Whitney C^2 topology by replacing f and π with $f \circ \pi'$ and $\pi'^{-1} \circ \pi$ for a C^∞ approximation π' of π in the Whitney C^2 topology. Hence by lemma 4.2 and properness of f and g, there

exists a C^∞ diffeomorphism π'' of M close to id in the Whitney C^2 topology such that $\pi''(Y) = X$. Replace f and π, once more, by $f \circ \pi''$ and $\pi''^{-1} \circ \pi$. Then we can assume that, moreover, $X = Y$. We want to modify π to be of class C^∞ on a neighborhood of X. Set $B(\epsilon) = \{x \in \mathbf{R}^n : |x| \le \epsilon\}$ for $\epsilon > 0 \in \mathbf{R}$. Let $\{U_i\}$ and $\{U'_i\}$ be locally finite open coverings of X in M such that $\overline{U'_i} \subset U_i$, such that $\pi(\overline{U'_i}) \subset U_i$, each $f|_{U_i}$ is C^∞ right equivalent to the function $\prod_{j=1}^n x_j^{\alpha_j} + \text{constant}$, for $x = (x_1, ..., x_n) \in \operatorname{Int} B(\epsilon_i)$ and for some $\epsilon_i > 0 \in \mathbf{R}$ and some $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbf{N}^n$ depending on i with $\alpha_1 > 0, ..., \alpha_{n'} > 0, \alpha_{n'+1} = \cdots \alpha_n = 0$ and that $U_i \cap X$ and U'_i are carried to $\operatorname{Int} B(\epsilon_i) \cap \{x_1 \cdots x_{n'} = 0\}$ and $B(\epsilon_i/2)$ by the diffeomorphism of equivalence. Then by induction on i it suffices to prove the following statement (for simplicity of notation we assume that $\epsilon_i = 3$ and $\overline{U'_i}$ is carried to $B(1)$).

Let C be a closed subset of $B(3)$. Let f and g be C^∞ functions on $\mathbf{R}^n$ such that f is of the form $x^\alpha = \prod_{j=1}^n x_j^{\alpha_j}$ for the above α and g is of the form $x^\alpha g'$ for some positive C^∞ function g' on $\mathbf{R}^n$. Let π be a C^2 embedding of $B(3)$ into $\mathbf{R}^n$ such that $f \circ \pi = g$ on $B(3)$ and $\pi(X \cap B(3)) \subset X$ where $X = \{x^\alpha = 0\}$. Let $\tau : B(3) \to \mathbf{R}^n$ be a C^2 approximation of π in the C^1 topology such that $\tau(X \cap B(3)) \subset X$, such that $f \circ \tau = g$ on a neighborhood of C in $B(3)$ and τ is of class C^∞ there. Then, fixing on $(B(3) - B(2)) \cup C$, we can approximate τ by a C^2 embedding $\tilde{\tau} : B(3) \to \mathbf{R}^n$ in the C^1 topology so that $\tilde{\tau}(X \cap B(3)) \subset X$, so that $f \circ \tilde{\tau} = g$ on $B(1)$ and $\tilde{\tau}$ is of class C^∞ on $B(1)$.

We prove the statement. Set $\tau(x) = (\tau_1(x), ..., \tau_n(x))$. Then $\tau_j(x)$ for each $1 \le j \le n'$ is divisible by x_j, to be precise, there exists a positive C^1 function F_j on $B(3)$ such that $\tau_j(x) = x_j F_j(x)$ since $\pi(X \cap B(3)) \subset X$ and $X = \{0\} \times \mathbf{R}^{n-1} \cup \cdots \cup \mathbf{R}^{n'-1} \times \{0\} \times \mathbf{R}^{n-n'}$ and π is close to id. The required approximation $\tilde{\tau} = (\tilde{\tau}_1, ..., \tilde{\tau}_n)$ also has to have the form $(x_1 \tilde{F}_1, ..., x_{n'} \tilde{F}_{n'}, \tilde{\tau}_{n'+1}, ..., \tilde{\tau}_n)$ for some positive C^1 functions $\tilde{F}_j$ and C^2 functions $\tilde{\tau}_{n'+1}, ..., \tilde{\tau}_n$. Set $F = (F_1, ..., F_{n'})$ and $\tilde{F} = (\tilde{F}_1, ..., \tilde{F}_{n'})$. Then F is of class C^∞ on a neighborhood of C, the condition $f \circ \tilde{\tau} = g$ on $B(1)$ coincides with the one $\tilde{F}^\alpha = g'$ on $B(1)$, and the other conditions which $\tilde{F}, \tilde{\tau}_{n'+1}, ..., \tilde{\tau}_n$ satisfy are that $\tilde{F} = F$ on $(B(3) - B(2)) \cup C$, that $(\tilde{F}, \tilde{\tau}_{n'+1}, ..., \tilde{\tau}_n)$ is an approximation of $(F, \tau_{n'+1}, ..., \tau_n)$ in the C^1 topology and that $\tilde{\tau}$ is of class C^2 on $B(3)$ and of class C^∞ on $B(1)$.

Set $Z = \{(x, y) \in B(3) \times \mathbf{R}^{n'} : y^\alpha = g'(x)\}$, which is a C^∞ submanifold with boundary of $B(3) \times \mathbf{R}^{n'}$ by the implicit function theorem since g' is positive. Note that $\tilde{F}^\alpha = g'$ on $B(1)$ if and only if $\operatorname{graph} \tilde{F}|_{B(1)} \subset Z$ and that $\operatorname{graph} F|_C \subset Z$. We can construct a C^∞ projection $p : W \to Z$ of a tubular neighborhood of Z in $B(3) \times \mathbf{R}^{n'}$ such that $p(x, y)$ for $(x, y) \in W$ is of the form $(x, p_2(x, y))$ as follows. Since g' is positive, $Z \cap \{x\} \times \mathbf{R}^{n'}$ for each $x \in B(3)$ is smooth and, moreover, the restriction to Z of the projection $B(3) \times \mathbf{R}^{n'} \to B(3)$ is submersive. Hence if we define $p(x, y)$ for each $(x, y) \in B(3) \times \mathbf{R}^{n'}$ near Z to be the orthogonal projection image of (x, y) to $Z \cap \{x\} \times \mathbf{R}^{n'}$, then p satisfies the requirements. Let $(\hat{F}, \hat{\tau}_{n'+1}, ..., \hat{\tau}_n)$ be a C^∞ approximation of $(F, \tau_{n'+1}, ..., \tau_n)$ in the C^1 topology, fixed on a neighborhood of C, and ϕ a C^∞ function on $B(3)$ such that $0 \le \phi \le 1$, $\phi = 1$ on $B(1)$ and $\phi = 0$ on $B(3) - B(2)$. Define a C^2 map $\tilde{F} = (\tilde{F}_1, ..., \tilde{F}_{n'}) : B(3) \to \mathbf{R}^{n'}$ by

$$\tilde{F}(x) = \phi(x) p_2(x, \hat{F}(x)) + (1 - \phi(x)) F(x) \quad \text{for } x \in B(3),$$

and set $\tilde{\tau} = (x_1 \tilde{F}_1, ..., x_{n'} \tilde{F}_{n'}, \hat{\tau}_{n'+1}, ..., \hat{\tau}_n)$ on $B(3)$. Then $\operatorname{graph} \tilde{F}|_{B(1)}$ is included

in Z because $\tilde F|_{B(1)}$ coincides with the map : $B(1)\ni x\to p_2(x,\tilde F(x))\in\mathbf{R}^{n'}$ whose graph is contained in Z; then $\tilde F=F$ on $B(3)-B(2)$ since $\phi=0$ there; then $\tilde F=F$ on C since $\hat F=F$ there and since $p(x,F(x))=(x,F(x))$ there; then $(\tilde F,\hat\tau_{n'+1},...,\hat\tau_n)$ is an approximation of $(F,\tau_{n'+1},...,\tau_n)$ in the C^1 topology since so is $(\hat F,,\hat\tau_{n'+1},...,\hat\tau_n)$; then $\tilde\tau$ is of class C^2 because if we set $p_2(x,y)=(p_{2,1}(x,y),...,p_{2,n'}(x,y))$ then

$$\tilde\tau_j(x)=\phi(x)x_jp_{2,j}(x,\hat F(x))+(1-\phi(x))\tau_j(x),\ 1\le j\le n';$$

finally $\tilde\tau$ is of class C^∞ on $B(1)$ since $\tilde F(x)=p_2(x,\hat F(x))$ on $B(1)$. Thus the statement is proved.

In conclusion, for some closed neighborhood V of $f(\operatorname{Sing} f)$ in $\mathbf{R}$ each of whose connected components contains one point of $f(\operatorname{Sing} f)$, there exists a C^2 diffeomorphism τ of M sufficiently close to π in the Whitney C^1 topology such that τ is of class C^∞ on $f^{-1}(V)$ and $f\circ\tau=g$ on $f^{-1}(V)$. Then the restrictions of f and g to $f^{-1}(\overline{\mathbf{R}-V})$ are proper and locally trivial maps onto $\overline{\mathbf{R}-V}$, moreover $f\circ\pi=g$ on $f^{-1}(\overline{\mathbf{R}-V})$ and $\tau|_{f^{-1}(\overline{\mathbf{R}-V})}$ is an approximation of $\pi|_{f^{-1}(\overline{\mathbf{R}-V})}$ in the Whitney C^1 topology. Hence we can modify τ so that $f\circ\tau=g$ and τ is of class C^∞ everywhere fixing on $f^{-1}(V)$. Therefore, f and g are C^∞ right equivalent, which proves (2).

Proof of (3). Let $0\ll l\in\mathbf{N}$. We prove first that f and g are semialgebraically C^l right equivalent and later that semialgebraic C^l right equivalence implies Nash right equivalence. We proceed with the former step as in the above proof of (2). Let π be a semialgebraic C^2 diffeomorphism of M such that $f\circ\pi=g$, and set $X=f^{-1}(f(\operatorname{Sing} f))$ and $g^{-1}(g(\operatorname{Sing} g))$. Let π' be a Nash approximation of π in the semialgebraic C^2 topology (Approximation Theorem I). Then π' is a diffeomorphism of M and $\pi'^{-1}\circ\pi$ is a semialgebraic C^2 approximation of id in the semialgebraic C^2 topology. Hence by replacing f and π with $f\circ\pi'$ and $\pi'^{-1}\circ\pi$, we assume that π is close to id in the semialgebraic C^2 topology. Moreover, we suppose that $X=Y$ as in the proof of (2) by using lemma 4.3 and its remark in place of lemma 4.2. Furthermore, by using lemma 4.6 we can reduce the problem to the case where M is the interior of a compact Nash manifold possibly with boundary M_1 and for each $x\in\partial M_1$, the germ (M_{1x},X_x) is Nash diffeomorphic to the germ at 0 of $(\mathbf{R}^{n-1}\times[0,\infty),\{(x_1,...,x_{n-1})\in\mathbf{R}^{n-1}:x_1\cdots x_{n'}=0\}\times(0,\infty))$ for some $n'\,(<n)\in\mathbf{N}$.

We modify π on a semialgebraic neighborhood of X. By lemma 4.7 and proposition 4.8,(iii) there exist **finite** open semialgebraic coverings $\{U_i\}$ and $\{U'_i\}$ of X in M such that the closure $\overline{U'_i}$ in M is contained in U_i, such that $\pi(\overline{U'_i})$ is contained in U_i, such that $f|_{U_i}$ is Nash right equivalent to $x^\alpha+$constant on $\operatorname{Int} B_{\xi_i}(\epsilon_i)$ where $\alpha=(\alpha_1,...,\alpha_n)\in\mathbf{N}^n$ depending on i with $\alpha_1>0,...,\alpha_{n'}>0,\alpha_{n'+1}=\cdots\alpha_n=0$, for $n'\,(<n)\in\mathbf{N}-\{0\}$ and $B_{\xi_i}(\epsilon_i)=\{x=(x_1,...,x_n)\in\mathbf{R}^n:x_n>0,|x^\alpha|\le\xi_i(x_n),|x|\le\epsilon_i\}$ for some $\epsilon_i>0\in\mathbf{R}$ and some positive Nash function ξ_i on $(0,\infty)$ and that $U_i\cap X$ and U'_i are carried to $\operatorname{Int} B_{\xi_i}(\epsilon_i)\cap\{x_1\cdots x_{n'}=0\}$ and $\operatorname{Int} B_{\xi_i/2}(\epsilon_i/2)$ by the diffeomorphism of equivalence. For modification of π on a semialgebraic neighborhood of X we need the following statement. Let $l'\in\mathbf{N}$ such that $l\le l'\le l+\#\{i\}$.

Let ξ be a small positive Nash function on $(0,\infty)$, and C a closed semialgebraic subset of $B_{3\xi}(3)$. Let f and g be Nash functions on $B_{4\xi}(4)$ such that f is of the form x^α for the above α and g is of the form $x^\alpha g'$ for some positive Nash function g' on

$B_{4\xi}(4)$. Let π be a semialgebraic C^2 embedding of $B_{3\xi}(3)$ into $B_{4\xi}(4)$ close to id in the semialgebraic C^2 topology such that $f\circ\pi = g$ on $B_{3\xi}(3)$ and $\pi(X\cap B_{3\xi}(3)) \subset X$ where $X = \{(x_1, ..., x_n) \in \mathbf{R}^n : x_1 \cdots x_{n'} = 0\}$. Let $\tau : B_{3\xi}(3) \to B_{4\xi}(4)$ be a semialgebraic C^2 approximation of π in the semialgebraic C^1 topology such that $\tau(X \cap B_{3\xi}(3)) \subset X$, such that $f \circ \tau = g$ on a closed semialgebraic neighborhood V of C in $B_{3\xi}(3)$ and τ is of class $C^{l'}$ there. Then, fixing on $(B_{3\xi}(3) - B_{2\xi}(2)) \cup C$ we can approximate τ by a semialgebraic C^2 embedding $\tilde{\tau} : B_{3\xi}(3) \to B_{4\xi}(4)$ in the semialgebraic C^1 topology so that $\tilde{\tau}(X \cap B_{3\xi}(3)) \subset X$, such that $f \circ \tilde{\tau} = g$ on $B_\xi(1)$ and $\tilde{\tau}$ is of class $C^{l'-1}$ on $B_\xi(1)$.

We prove the statement. As before, set $\tau = (\tau_1, ..., \tau_n)$, $\tilde{\tau} = (\tilde{\tau}_1, ..., \tilde{\tau}_n)$, let F_j and $\tilde{F}_j$, $1 \le j \le n'$, be positive semialgebraic C^1 functions on $B_{3\xi}(3)$ such that $\tau_j = x_j F_j(x)$ and $\tilde{\tau}_j = x_j \tilde{F}_j(x)$ on $B_{3\xi}(3)$, and set $F = (F_1, ..., F_{n'})$ and $\tilde{F} = (\tilde{F}_1, ..., \tilde{F}_{n'})$. Note that F_j are of class $C^{l'-1}$ on a semialgebraic neighborhood of C in $B_{3\xi}(3)$, which is different to F_j in the proof of (2) where they are of class C^∞. Then the required conditions are that $\tilde{F}^\alpha = g'$ on $B_\xi(1)$, that $\tilde{F} = F$ on $(B_{3\xi}(3) - B_{2\xi}(2)) \cup C$, that $(\tilde{F}, \tilde{\tau}_{n'+1}, ..., \tilde{\tau}_n)$ is a semialgebraic C^1 approximation of $(F, \tau_{n'+1}, ..., \tau_n)$ in the semialgebraic C^1 topology, and $\tilde{\tau}$ is of class C^2 on $B_{3\xi}(3)$ and of class $C^{l'-1}$ on $B_\xi(1)$.

Set $Z = \{(x, y) \in B_{3\xi}(3) \times \mathbf{R}^{n'} : y^\alpha = g'(x)\}$, which is a Nash submanifold with boundary of $B_{3\xi}(3) \times \mathbf{R}^{n'}$, and let $p : W \to Z$ be a Nash projection of a semialgebraic tubular neighborhood of Z in $B_{3\xi}(3) \times \mathbf{R}^{n'}$ such that $p(x, y)$ for $(x, y) \in W$ is of the form $(x, p_2(x, y))$, which is constructed as before. Let $(\hat{F}, \hat{\tau}_{n'+1}, ..., \hat{\tau}_n)$ be a Nash approximation of $(F, \tau_{n'+1}, ..., \tau_n)$ in the semialgebraic C^1 topology, and ϕ and ψ semialgebraic $C^{l'}$ functions on $B_{3\xi}(3)$ such that $0 \le \phi \le 1$, such that $\phi = 1$ on $B_\xi(1)$ and $\phi = 0$ on $B_{3\xi}(3) - B_{2\xi}(2)$, such that $0 \le \psi \le 1$ and $\psi = 1$ on $B_{3\xi}(3) - V$ whereas $\psi = 0$ on a semialgebraic neighborhood of C in $B_{3\xi}(3)$ smaller than $\operatorname{Int} V$. Set

$$\tilde{F}(x) = \phi(x) p_2\big(x, \psi(x)\hat{F}(x) + (1 - \psi(x))F(x)\big) + (1 - \phi(x))F(x) \quad \text{for } x \in B_{3\xi}(3),$$

and
$$\tilde{\tau} = (x_1\tilde{F}_1, ..., x_{n'}\tilde{F}_{n'}, \hat{\tau}_{n'+1}, ..., \hat{\tau}_n) \quad \text{on } B_{3\xi}(3).$$

Then we see as before that the required conditions are satisfied. Hence the statement is proved.

By the statement, a partition of unity of class semialgebraic C^l and by remark 2.11,(5)$'$ we obtain an open semialgebraic neighborhood U of X and a semialgebraic C^2 diffeomorphism τ of M close to π in the semialgebraic C^1 topology such that τ is of class C^l on U and $f \circ \tau = g$ on U (the point is that after fixing U we can choose τ so as to be arbitrarily close to id). Then we modify τ so that τ is of class semialgebraic C^l and $f \circ \tau = g$, i.e., f and g are semialgebraically C^l right equivalent as follows.

Let η be a semialgebraic C^l function on M such that $0 \le \eta \le 1$, such that $\eta = 0$ outside of U and $\eta = 1$ on a smaller semialgebraic neighborhood of X, and set $A = \{(x, y) \in (M - X)^2 : f(y) = g(x)\}$. Then A is a Nash manifold and there exists a Nash projection $q : Q \to A$ of a small semialgebraic tubular neighborhood of A in the square of the ambient Euclidean space of M of the form $q(x, y) = (x, q_2(x, y))$ for $x \in M - X$. Let $\check{\tau}$ be a Nash approximation of τ in the semialgebraic C^1 topology, and set

$$\check{\check{\tau}} = q_2\big(x, \eta(x)\tau(x) + (1 - \eta(x))\check{\tau}(x)\big) \quad \text{for } x \in M$$

Then $\check{\check{\tau}}$ is well-defined because the graph of the map from M to the ambient Euclidean space of $M : x \to \eta(x)\tau(x) + (1-\eta(x))\check{\tau}(x)$ is contained in Q, hence $\check{\check{\tau}}$ is a semialgebraic C^l diffeomorphism of M and $f \circ \check{\check{\tau}} = g$. Thus the former step of the proof is achieved.

Let $0 \ll l^{(3)} \ll \cdot\cdot \ll l \in \mathbf{N}$. For the latter step also we can assume that $X = Y$ and that there exists a semialgebraic C^l diffeomorphism π of M close to id in the semialgebraic C^l topology such that $f \circ \pi = g$. Let μ be a Nash function on $\mathbf{R}$ such that $\mu^{-1}(0) = f(\operatorname{Sing} f)$ and μ is regular at $\mu^{-1}(0)$. Consider $\mu \circ f$ and $\mu \circ g$. Their zero sets are X, they have only normal crossing singularities at X, the same sign at each point of M and the same multiplicity at each point of X, and we see easily that the Nash function on M, defined to be $\mu \circ g / \mu \circ f$ on $M - X$, is close to 1 in the semialgebraic C^l topology. Hence the conditions in lemma 4.7 are satisfied and there exists a Nash diffeomorphism π' of M close to id in the semialgebraic $C^{l'}$ topology such that $\pi'(X) = X$ and $f \circ \pi' - g$ is l'-flat at X. Thus, replacing f and π with $f \circ \pi'$ and $\pi'^{-1} \circ \pi$, we assume that $f - g$ is l'-flat at X and π is close to id in the semialgebraic $C^{l'}$ topology.

By proposition 4.9 we can assume that M is the interior of a compact Nash manifold possibly with corners M_1 and f is the restriction to M of a Nash function f_1 on M_1 with only normal crossing singularities. Then by the definition of semialgebraic C^l topology, π is extensible to a semialgebraic $C^{l'}$ diffeomorphism π_1 of M_1 such that $\pi_1 - \mathrm{id}$ is l'-flat at ∂M_1. Hence g also is extensible to a semialgebraic $C^{l'}$ function g_1 on M_1, and $f_1 - g_1$ is close to 0 in the $C^{l'}$ topology and l'-flat at ∂M_1. Let v_i, for $i = 1, \dots, N$, be Nash vector fields on M_1 spanning the tangent space of M_1 at each point, ν_1 a non-negative Nash function on M_1 with zero set ∂M_1 and regular there, and set $\nu_2 = \sum_{i=1}^N (v_i f_1)^2$ and $\nu = \nu_1^{l''} \nu_2$. Then the radical of $\nu_2 \mathcal{N}$ is the sheaf of $\mathcal{N}$-ideals defined by $X \cup \partial M_1$, and $f_1 - g_1$ is divisible by ν; to be precise, there exists a semialgebraic $C^{l''}$ function β on M_1 such that $f_1 - g_1 = \nu\beta$. Moreover, β is close to 0 in the $C^{l^{(3)}}$ topology. Actually, by lemma 2.12 the map $C^\infty(M_1) \ni h \to \nu h \in \nu C^\infty(M_1)$ is open. Hence for $h \in C^\infty(M_1)$, if νh is close to 0 in the $C^{l''}$ topology then h is close to 0 in the $C^{l^{(3)}}$ topology. This holds for $h \in C^{l''}(M_1)$ also because h of class $C^{l''}$ is approximated by a C^∞ function h' in the $C^{l''}$ topology and $\nu \cdot (h - h')$ and hence $\nu h'$ are close to 0 in the $C^{l''}$ topology. Therefore, β is close to 0 in the $C^{l^{(3)}}$ topology.

It follows from the definition of semialgebraic C^l topology that $\nu_1^{l''} \beta|_M$ is close to 0 in the semialgebraic $C^{l^{(3)}}$ topology. Then the conditions in proposition 4.8,(ii) for f and $g\, (= f - \nu_1^{l''} \beta \sum_{i=1}^N (v_i f_1)^2|_M)$ are satisfied. Hence f and g are Nash right equivalent.

We can prove the case with corners in the same way. $\square$

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